HOWE PAIRS, SUPERSYMMETRY, AND RATIOS OF RANDOM CHARACTERISTIC POLYNOMIALS FOR THE UNITARY GROUPS U_N

by

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Abstract. — For the classical compact Lie groups $K \equiv \mathrm{U}_N$ the autocorrelation functions of ratios of characteristic polynomials $(z,w) \mapsto \mathrm{Det}(z-k)/\mathrm{Det}(w-k)$ are studied with $k \in K$ as random variable. Basic to our treatment is a property shared by the spinor representation of the spin group with the Shale-Weil representation of the metaplectic group: in both cases the character is the analytic square root of a determinant or the reciprocal thereof. By combining this fact with Howe's theory of supersymmetric dual pairs $(\mathfrak{g}, K)$, we express the K-Haar average product of p ratios of characteristic polynomials and q conjugate ratios as a character χ which is associated with an irreducible representation of the Lie superalgebra $\mathfrak{g} = \mathfrak{gl}_{n|n}$ for $n = p+q$. The primitive character χ is shown to extend to an analytic radial section of a real supermanifold related to $\mathfrak{gl}_{n|n}$, and is computed by invoking Berezin's description of the radial parts of Laplace-Casimir operators for $\mathfrak{gl}_{n|n}$. The final result for χ looks like a natural transcription of the Weyl character formula to the context of highest-weight representations of Lie supergroups.

While several other works have recently reproduced our results in the stable range $N \geq \max(p,q)$, the present approach covers the *full* range of matrix dimensions $N \in \mathbb{N}$.

To make this paper accessible to the non-expert reader, we have included a chapter containing the required background material from superanalysis.

Contents

1. Introduction

Let K be one of the compact Lie groups U_N, O_N, or USp_N, and let K be equipped with Haar measure dk normalized by $\int_K dk = 1$. We are interested in products of ratios of characteristic polynomials $\mathrm{Det}(z-k)$, averaged with respect to the probability measure dk. Our goal is to derive exact expressions for such autocorrelation functions.

Quantities of a similar kind, for Gaussian distributed Hermitian matrices, were first computed some time ago in the large-N limit by Andreev and Simons [**1**], using an elaboration of the supersymmetry method pioneered by Wegner and Efetov. Then, in 1996-97, one of us developed a variant [**21**] of the Wegner-Efetov method to handle the case of Haar-ensemble averages for the compact Lie groups above. With guidance from a perceptive remark of Sarnak [**19**], it soon transpired that the mathematical foundation of that variant had existed in classical invariant theory for decades, and is known there as *Howe duality*. The main ideas – namely, Howe dual pairs acting in the *spinor-oscillator* module, the formula for the spinor-oscillator character, and the fact that the K-Haar autocorrelation can be regarded as the character associated with a *highest-weight representation* of some *Lie superalgebra* $\mathfrak{g}$ – were presented at the MSRI workshop on "Random matrices and their Applications" (Berkeley, June 1999).

The second step of that original approach was to express the primitive $\mathfrak{g}$-character as a co-adjoint orbit integral (actually, a Berezin superintegral) á la Kirillov, and finally to compute this integral by a super version of the localization principle for equivariantly closed differential forms [**5**]. While this last step works in the so-called *stable range*, i.e., for large enough N, the superintegral becomes too subtle an object to handle for small values of N and, in any case, this type of analysis has not yet been perfected to the stage of yielding mathematical theorems. In the present paper, we replace it by a combination of algebraic and analytical techniques *avoiding* superintegration.

While this research was initially motivated by quantum chaos and the physics of disordered systems, the main motivation for bringing it to completion comes from analytic number theory. In that area, random matrices have been a recurrent and growing theme ever since Montgomery conjectured the pair correlation function of the non-trivial zeroes of the Riemann zeta function to be asymptotic to Dyson's pair correlation function for the Gaussian Unitary Ensemble (GUE) and since Odlyzko's high-precision numerical study demonstrated asymptotic agreement of the distribution of spacings between the Riemann zeroes with the spacing distribution predicted from GUE.

The conjectured relation between the zeroes of zeta functions and eigenvalues of random matrices was much clarified by the work of Katz and Sarnak [**16**], which shifted the emphasis from Riemann zeta to *families* of L-functions. Carried out in the setting of L-functions over finite fields, this work explained how a limit procedure for families leads to equidistribution on one of the classical compact Lie groups SU_N, U_N, SO_{2N+1}, USp_N, or SO_{2N}, the group being determined by the symmetry of the family.

Keating and Snaith made the proposal that the connection between L-functions and random matrices should carry over to the case of global L-functions. In particular, they conjectured [**17**] that the critical value distribution of the Riemann zeta function (or, more precisely, a certain universal part thereof, which is hard to access by number

theory methods) can be modelled by the characteristic polynomial of a random matrix $u \in \mathrm{U}_N$. A sizable number of precise generalizations of this proposal have since been made, although most of the theory to date remains at the conjectural level.

In a companion paper [**9**] we present various conjectures that are suggested for L-functions by the Keating-Snaith conjecture and the results of our work. The present paper is solely concerned with the random-matrix aspect of this unfolding story; the reader is referred to our second paper for the related details concerning L-functions.

1.1. Statement of result. — We now define the quantity to be studied in this paper. Postponing the cases of the compact Lie groups $K = \mathrm{O}_N$ and $K = \mathrm{USp}_{2N}$ to another publication we focus on the case of $K = \mathrm{U}_N$ and, fixing a pair of positive integers p and q, consider for $u \in \mathrm{U}_N$ the product of ratios of characteristic polynomials

$$Z(t,u) = \mathrm{e}^{\lambda_N} \prod_{j=1}^{p} \frac{\mathrm{Det}(\mathrm{Id}_N - \mathrm{e}^{\mathrm{i}\psi_j}\, u)}{\mathrm{Det}(\mathrm{Id}_N - \mathrm{e}^{\phi_j}\, u)} \prod_{l=p+1}^{p+q} \frac{\mathrm{Det}(\mathrm{Id}_N - \mathrm{e}^{-\mathrm{i}\psi_l}\, \bar{u})}{\mathrm{Det}(\mathrm{Id}_N - \mathrm{e}^{-\phi_l}\, \bar{u})}\,, \tag{1.1}$$

where $\bar{u}$ is the complex conjugate of the unitary matrix u. The parameters $\psi_1, \ldots, \phi_{p+q}$ are complex, with the range of the ϕ_k restricted by $\mathfrak{Re}\,\phi_j < 0 < \mathfrak{Re}\,\phi_l$ for $j = 1, \ldots, p$ and $l = p+1, \ldots, p+q$. It will be convenient to think of these parameters as defining a diagonal matrix $t \in \mathrm{End}(\mathbb{C}^{2(p+q)})$ by

$$t = \mathrm{diag}\,(\mathrm{e}^{\mathrm{i}\psi_1}, \ldots, \mathrm{e}^{\mathrm{i}\psi_{p+q}}, \mathrm{e}^{\phi_1}, \ldots, \mathrm{e}^{\phi_{p+q}})\,. \tag{1.2}$$

The multiplier e^{λ_N} is given by the linear combination of parameters

$$\lambda_N = N \textstyle\sum_{l=p+1}^{p+q} (\mathrm{i}\psi_l - \phi_l)\,. \tag{1.3}$$

Notice that a more concise form of (1.1) is

$$Z(t,u) = \prod_{k=1}^{n} \frac{\mathrm{Det}(\mathrm{Id}_N - \mathrm{e}^{\mathrm{i}\psi_k} u)}{\mathrm{Det}(\mathrm{Id}_N - \mathrm{e}^{\phi_k}\, u)} \qquad (n = p+q)\,. \tag{1.4}$$

Why do we exponentiate our parameters as $\mathrm{e}^{\mathrm{i}\psi_k}$ in the numerator as opposed to e^{ϕ_k} in the denominator? The answer is that at some point we will take advantage of a certain Riemannian structure, by restricting the complex parameters ψ_k and ϕ_k to $\mathbb{R}$.

If du denotes the U_N-Haar measure normalized by $\int_{\mathrm{U}_N} du = 1$, our goal is to establish an exact expression for the ensemble average

$$\chi(t) := \int_{\mathrm{U}_N} Z(t,u)\, du \tag{1.5}$$

for all matrix dimensions N. To state our result for χ, let S_{p+q} be the group of permutations of the $p+q$ objects

$$\psi_1\,, \ldots, \psi_p\,, \psi_{p+1}\,, \ldots, \psi_{p+q}\,.$$

Theorem 1.1. — *For all $N \in \mathbb{N}$ the Haar-ensemble average $\chi(t) = \int_{\mathrm{U}_N} Z(t,u)\,du$ is expressed by*

$$\chi(t) = \frac{1}{p!\,q!} \sum_{w \in \mathrm{S}_{p+q}} \prod_{l=p+1}^{p+q} \frac{\mathrm{e}^{N\,\mathrm{i}w(\psi_l)}}{\mathrm{e}^{N\phi_l}} \prod_{j=1}^{p} \frac{(1-\mathrm{e}^{\phi_j - \mathrm{i}w(\psi_l)})\,(1-\mathrm{e}^{\mathrm{i}w(\psi_j)-\phi_l})}{(1-\mathrm{e}^{\mathrm{i}w(\psi_j)-\mathrm{i}w(\psi_l)})\,(1-\mathrm{e}^{\phi_j-\phi_l})}\,. \tag{1.6}$$

Remark. — Let $\mathrm{S}_p \times \mathrm{S}_q$ be the subgroup of S_{p+q} which permutes the first p parameters $\psi_1, \ldots, \psi_p$ separately from the last q parameters $\psi_{p+1}, \ldots, \psi_{p+q}$. It is easy to see that the terms in the sum for χ do not change under the substitution $w \to ww'$ if $w' \in \mathrm{S}_p \times \mathrm{S}_q$. Therefore, without changing the result, one may replace the sum over $w \in \mathrm{S}_{p+q}$ by a sum over cosets $[w] \in \mathrm{S}_{p+q}/(\mathrm{S}_p \times \mathrm{S}_q)$ and drop the factor of $(p!\,q!)^{-1}$.

To give a smooth statement without restrictions on the values of N, we have taken the number of characteristic polynomials in the numerator and denominator of (1.1) to be equal. This, of course, is not a serious limitation; from formula (1.6) one immediately produces answers for the case of unequal numbers by sending one or several of the parameters $\mathrm{e}^{\pm\mathrm{i}\psi_k}$ or $\mathrm{e}^{\pm\phi_k}$ to zero on both sides of the equation. In fact, an easy induction (spelled out in Sect. 4.8) yields the following statement.

Corollary 1.2. — *If $p' \le p+N$ and $q' \le q+N$ then*

$$\int_{\mathrm{U}_N} \frac{\prod_{j=1}^{p} \mathrm{Det}(\mathrm{Id}_N - \mathrm{e}^{\mathrm{i}\psi_j}\,u) \prod_{l=p+1}^{p+q} \mathrm{Det}(\mathrm{Id}_N - \mathrm{e}^{-\mathrm{i}\psi_l}\,\bar{u})}{\prod_{j'=1}^{p'} \mathrm{Det}(\mathrm{Id}_N - \mathrm{e}^{\phi_{j'}} u) \prod_{l'=p'+1}^{p'+q'} \mathrm{Det}(\mathrm{Id}_N - \mathrm{e}^{-\phi_{l'}}\,\bar{u})}\,du$$

$$= \sum_{[w] \in \mathrm{S}_{p+q}/\mathrm{S}_p \times \mathrm{S}_q} \prod_{k=p+1}^{p+q} \frac{\mathrm{e}^{N\,\mathrm{i}w(\psi_k)}}{\mathrm{e}^{N\,\mathrm{i}\psi_k}} \times \frac{\prod_{j',l}(1-\mathrm{e}^{\phi_{j'}-\mathrm{i}w(\psi_l)}) \prod_{j,l'}(1-\mathrm{e}^{\mathrm{i}w(\psi_j)-\phi_{l'}})}{\prod_{j,l}(1-\mathrm{e}^{\mathrm{i}w(\psi_j)-\mathrm{i}w(\psi_l)}) \prod_{j',l'}(1-\mathrm{e}^{\phi_{j'}-\phi_{l'}})}\,.$$

When the matrix dimension N is sufficiently large, formula (1.6) can be obtained from a variety of different approaches. The present paper develops the point of view that (1.6) is best seen as a result in the theory of Lie superalgebras and supergroups: it is the latter framework that delivers the answer for *all* values of $N \ge 1$.

1.1.1. Howe duality. — The method we will use to prove Thm. 1.1 rests on the fact that U_N is in Howe duality with a Lie superalgebra $\mathfrak{gl}_{n|n}$ for $n = p+q$. The details are briefly speaking as follows. Let $V = V_0 \oplus V_1$ with $V_0 \simeq V_1$ be a $\mathbb{Z}_2$-graded vector space, where V_0 is the direct sum of p fundamental modules and q co-fundamental modules for U_N. Let $\mathscr{A}_V$ be the spinor-oscillator module of V, i.e., the tensor product of the exterior algebra of V_1 with the symmetric algebra of V_0:

$$\mathscr{A}_V = \wedge(V_1) \otimes \mathrm{S}(V_0)\,. \tag{1.7}$$

U_N acts on $\mathscr{A}_V$, and so does $\mathfrak{gl}_{n|n}$, and these two actions commute.

Now recall the definition of the diagonal parameter matrix t by (1.2), and think of its logarithm as lying in a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{gl}_{n|n}$. Then $t = \mathrm{e}^{\ln t}$ operates on $\mathscr{A}_V$ by the exponential of the action of $\mathfrak{gl}_{n|n}$ and, as will be explained in some pedagogical detail in Sect. 2, the supertrace of the operator representing the pair (t,u) on $\mathscr{A}_V$ equals

the product of ratios (1.1). Moreover, the joint action of the Howe dual pair $(\mathfrak{gl}_{n|n}, \mathrm{U}_N)$ on $\mathscr{A}_V$ is *multiplicity-free*, which leads to the following key observation.

Proposition 1.3. — *The function $\chi : t \mapsto \int_{\mathrm{U}_N} Z(t,u)\, du$ is a primitive $\mathfrak{gl}_{n|n}$-character, and is given by the $\mathfrak{gl}_{n|n}$-representation on the space of U_N-invariants in $\mathscr{A}_V$.*

From Prop. 1.3 (which summarizes Prop. 2.10), our task is to compute the primitive character $t \mapsto \chi(t)$. This is not entirely straightforward as the representation at hand is *atypical* and lies outside the range of applicability of known character formulas. Some answer for χ (in the form of a Littlewood-Richardson sum over products of $\mathfrak{gl}_{p|p}$- and $\mathfrak{gl}_{q|q}$-characters, using the $\mathfrak{gl}_{p+q|p+q} \to \mathfrak{gl}_{p|p} \times \mathfrak{gl}_{q|q}$ branching rules) has recently been given in [**6**]; that answer, however, is not of the explicit form we are looking for.

In this paper $\chi(t)$ will be computed explicitly by drawing on some of Berezin's results for the radial parts of the Laplace-Casimir operators for the classical *Lie supergroup* $\mathrm{U}_{n|n}$. The outcome is the formula stated in Thm. 1.1. Let us now rewrite that formula using the standard language of representation theory.

1.1.2. Representation-theoretic formulation. — It is appropriate to regard the $2n$ parameters $\psi_1, \ldots, \psi_n, \phi_1, \ldots, \phi_n$ as linear functions on a Cartan subalgebra $\mathfrak{h} \simeq \mathbb{C}^n \oplus \mathbb{C}^n$ of diagonal transformations in the Lie superalgebra $\mathfrak{g} \equiv \mathfrak{gl}_{n|n}$; by linearly combining them one can then form the roots of $\mathfrak{g}$ and the weights of its representations.

Roots $\alpha : \mathfrak{h} \to \mathbb{C}$ are defined as the eigenvalues of the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ as usual: $[H,X] = \alpha(H)X$. A root α is called *even* or *odd* depending on whether the corresponding eigenvector X is an even or odd element of the Lie superalgebra $\mathfrak{g}$.

The representation of $\mathfrak{g} = \mathfrak{gl}_{n|n}$ on the subspace of U_N-invariants in $\mathscr{A}_V$ will be seen to be a highest-weight representation, with the role of highest weight being played by the linear function $\lambda_N : \mathfrak{h} \to \mathbb{C}$ of (1.3). Our result for χ is to be expressed in terms of the λ_N-positive roots, i.e., the sets of even and odd roots of ($\mathfrak{h}$ acting on) $\mathfrak{g}$ which are positive with respect to the highest weight λ_N. These are the sets

$$\begin{aligned} \Delta^+_{\lambda,0} \quad &: \quad \mathrm{i}\psi_l - \mathrm{i}\psi_j\,, \quad \phi_l - \phi_j\,, \\ \Delta^+_{\lambda,1} \quad &: \quad \phi_l - \mathrm{i}\psi_j\,; \quad \mathrm{i}\psi_l - \phi_j \quad (1 \le j \le p < l \le n)\,. \end{aligned} \tag{1.8}$$

The primitive characters of a compact Lie group K are expressed by the Weyl character formula (see, e.g., [**18**]), which says that if $\rho : K \to \mathrm{U}(V_\lambda)$ is an irreducible representation with regular highest weight λ, then

$$\mathrm{Tr}_{V_\lambda}\, \rho(t) = \sum_{w \in W} \frac{\mathrm{e}^{w(\lambda)}}{\prod_{\alpha \in \Delta^+_\lambda} (1 - \mathrm{e}^{-w(\alpha)})} (\ln t)\,,$$

where W is the Weyl group of K, and Δ^+_λ denotes the system of λ-positive roots.

Our answer for χ will look like a natural generalization of this formula to the case of Lie supergroups. The Weyl group W of our problem is the same as the symmetric group S_n of permutations of the n basis vectors of the first summand in $\mathfrak{h} \simeq \mathbb{C}^n \oplus \mathbb{C}^n$ (which are dual to the coordinate functions $\psi_1, \ldots, \psi_n$). While the Weyl group W is

defined by its action on the Cartan subalgebra $\mathfrak{h}$, what is needed here is the induced action on linear functions $f \in \mathfrak{h}^*$ by $w(f) := f \circ w^{-1}$.

Because the highest weight λ_N of the $\mathfrak{g}$-representation at hand turns out to be non-regular, it is not W that matters but its quotient by the subgroup W_λ that fixes λ_N. This is the subgroup $W_\lambda \simeq \mathrm{S}_p \times \mathrm{S}_q$ stabilizing the decomposition $\mathbb{C}^n = \mathbb{C}^p \oplus \mathbb{C}^q$.

Of course every coset $[w] \in W/W_\lambda$ uniquely defines a transformed highest weight $w(\lambda_N)$. It also determines transformed sets of λ_N-positive roots $w(\Delta^+_{\lambda,0})$ and $w(\Delta^+_{\lambda,1})$ which are independent of the choice of representative w.

Given these definitions and the insight from Prop. 1.3, the statement of Thm. 1.1 can be rephrased as follows.

Proposition 1.4. — *For all $N \in \mathbb{N}$ the character $t \mapsto \chi(t)$ is given by a generalization of the Weyl character formula:*

$$\chi(t) = \sum_{[w]\in W/W_\lambda} \mathrm{e}^{w(\lambda_N)} \frac{\prod_{\beta\in\Delta^+_{\lambda,1}}(1-\mathrm{e}^{-w(\beta)})}{\prod_{\alpha\in\Delta^+_{\lambda,0}}(1-\mathrm{e}^{-w(\alpha)})}(\ln t)\,. \tag{1.9}$$

Remark. — In a companion paper [**15**] it is proved that a completely analogous result holds for the case of ratios of random characteristic polynomials for the compact Lie groups $K = \mathrm{USp}_{2N}$, $K = \mathrm{O}_N$, and $K = \mathrm{SO}_N$.

A number of further remarks on the scope of Thm. 1.1 are called for.

1.1.3. Compact sector. — Consider now removing all of the ϕ-parameters, i.e., send $\phi_j \to -\infty$ in $\mathrm{Det}(\mathrm{Id}_N - \mathrm{e}^{\phi_j} u)^{-1}$ and $\phi_l \to +\infty$ in $\mathrm{Det}(\mathrm{Id}_N - \mathrm{e}^{-\phi_l}\,\overline{u})^{-1}$. Then the formula of Thm. 1.1 reduces to

$$\begin{aligned} &\int\limits_{\mathrm{U}_N} \prod_{j=1}^{p} \mathrm{Det}(\mathrm{Id}_N - \mathrm{e}^{\mathrm{i}\psi_j} u) \prod_{l=p+1}^{p+q} \mathrm{Det}(\mathrm{e}^{\mathrm{i}\psi_l}\mathrm{Id}_N - \overline{u})\, du \\ &= \sum_{[w]\in \mathrm{S}_{p+q}/\mathrm{S}_p\times\mathrm{S}_q} \prod_{l=p+1}^{p+q} \mathrm{e}^{N\mathrm{i}w(\psi_l)} \prod_{j=1}^{p} \frac{1}{1-\mathrm{e}^{\mathrm{i}w(\psi_j)-\mathrm{i}w(\psi_l)}}\,, \end{aligned} \tag{1.10}$$

which is a result that can also be extracted from [**7**]. The mathematical situation in this limit is ruled by a Howe dual pair [**13**] of compact Lie groups

$$K' \times K = \mathrm{U}_{p+q} \times \mathrm{U}_N\,;$$

which is to say that $K' \times K$ acts without multiplicity on an exterior algebra $\wedge(V_1)$, and the left-hand side of (1.10) is the character of the irreducible representation of K' on the subalgebra of K-invariants in $\wedge(V_1)$. The right-hand side of (1.10) is nothing but the classical Weyl character formula evaluated for the irreducible U_{p+q}-representation with (non-regular) highest weight

$$\lambda'_N = N \textstyle\sum_{l=p+1}^{p+q} \mathrm{i}\psi_l\,. \tag{1.11}$$

1.1.4. Non-compact sector. — When all of the ψ-parameters are removed, Thm. 1.1 reduces to a statement about the autocorrelations of *reciprocals* of characteristic polynomials. This particular limit is ruled by the Howe dual pair

$$G \times K = \mathrm{U}_{p,q} \times \mathrm{U}_N \ ,$$

where $\mathrm{U}_{p,q}$ is the non-compact group of isometries of the pseudo-unitary vector space $\mathbb{C}^p \oplus \mathbb{C}^q$. The pair $G \times K$ acts without multiplicity on a symmetric algebra $\mathrm{S}(V_0)$ (generated by a direct sum of p copies of $\mathbb{C}^N$ and q copies of the dual vector space of $\mathbb{C}^N$) by the *Shale-Weil* or *oscillator representation*.

Taking a product of $p+q$ reciprocals of characteristic polynomials and their conjugates and averaging with Haar measure of K, one again gets a primitive character:

$$\int_{\mathrm{U}_N} \prod_{j=1}^{p} \mathrm{Det}(\mathrm{Id}_N - \mathrm{e}^{\phi_j} u)^{-1} \prod_{l=p+1}^{p+q} \mathrm{Det}(\mathrm{e}^{\phi_l} \mathrm{Id}_N - \bar{u})^{-1} \, du =: \chi''(t_0) \ ; \tag{1.12}$$

which is the character $\chi''(t_0) \equiv \chi''(\mathrm{e}^{\phi_1}, \ldots, \mathrm{e}^{\phi_{p+q}})$ of the representation of G on the subalgebra of K-invariants in $\mathrm{S}(V_0)$. For all positive integers p, q, and N, these G-representations are irreducible, unitary, and with highest weight

$$\lambda_N'' = -N \textstyle\sum_{l=p+1}^{p+q} \phi_l \ . \tag{1.13}$$

1.1.5. Stable versus non-stable range. — In the range $N \geq p+q$, the representations with highest weight (1.13) belong to the *principal discrete series* of $G = \mathrm{U}_{p,q}$, which is to say that they can be realized by Hilbert spaces of L^2-functions on G and are covered by an analog of the Borel-Weil correspondence known for the compact case. The characters in that range turn out to have a particularly simple expression:

$$\chi''(\mathrm{e}^{\phi_1}, \ldots, \mathrm{e}^{\phi_{p+q}}) = \mathrm{e}^{\lambda_N''} \prod_{j=1}^{p} \prod_{l=p+1}^{p+q} \frac{1}{1 - \mathrm{e}^{\phi_j - \phi_l}} \ . \tag{1.14}$$

This formula actually continues to hold as long as $N \geq \max(p,q)$. As a matter of fact, in that range the algebra of K-invariants in the oscillator module $\mathrm{S}(V_0)$ is *freely generated* by the quadratic invariants that correspond to the system of λ_N''-positive roots.

The range $N \geq \max(p,q)$ where (1.14) holds is called the *stable* range. From the vantage point of (1.6) the simplicity of the result (1.14) comes about because throughout the stable range, only the term from the identity coset $[e]$ in the sum over $[w] \in \mathrm{S}_{p+q}/(\mathrm{S}_p \times \mathrm{S}_q)$ survives the limit of removing the parameters ψ_k.

On the other hand, the said characters *outside* the stable range are of a different nature. Their representations have no realization by L^2-functions on G and are called *singular*. The algebra of K-invariants outside the stable range is *not* freely generated; there exist relations and the characters therefore do not have any expression as simple as (1.14). In fact, when these characters are extracted from our supersymmetric master formula (1.6), there is a build up of combinatorial complexity because the product over odd-root factors $(1 - \mathrm{e}^{\phi_j - \mathrm{i}w(\psi_l)})(1 - \mathrm{e}^{\mathrm{i}w(\psi_j) - \phi_l})$ starts competing with the highest weight factor $\mathrm{e}^{N \sum \mathrm{i}w(\psi_l)}$ so that an increasing number of terms survives the process of removing the ψ-parameters.

1.1.6. Other approaches. — Given the existence of a stable and a "non-stable" range for the non-compact sector, one should expect the same distinction to be visible also in the full, supersymmetric situation. This is indeed the case.

Our answer (1.6) has recently been reproduced partially by two independent approaches: by the work of Conrey, Forrester and Snaith [**8**] which is based on the results for orthogonal polynomials by Fyodorov and Strahov [**11**] and Baik, Deift and Strahov [**4**], and in a preprint by Bump and Gamburd [**3**] using symmetric function theory. Both make a statement only in the stable range for N. A fourth viable approach in that range is our original 1999 method of computation (i.e., expressing the character in question as a co-adjoint orbit superintegral and then calculating it by an equivariant localization technique) which is how the formulas (1.9) and (1.6) were first obtained.

From our experience, proving Thm. 1.1 *outside* the stable range for N is qualitatively more difficult! In fact, in view of the singular nature of the representations in the underlying non-compact situation, we consider it somewhat of a miracle that a formula as universal and simple as (1.6) happens to be true for all $N \in \mathbb{N}$. The present paper owes its length to our desire to give a transparent and self-contained proof of it.

1.2. Some basic definitions of superanalysis. — Let $V = V_1 \oplus V_0$ be a $\mathbb{Z}_2$-graded vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. An element $v \in V$ is called *odd* if $v \in V_1 \setminus \{0\}$ and *even* if $v \in V_0 \setminus \{0\}$. Elements which are either even or odd are called *homogeneous*. V comes with a *parity function* $|\cdot|$ which is defined on homogeneous elements and takes the value $|v| = 1$ for v odd and $|v| = 0$ for v even. If $V_1 = \mathbb{K}^p$ and $V_0 = \mathbb{K}^q$, one also writes $V = \mathbb{K}^{p|q}$. Depending on the context we will sometimes write $V = V_0 \oplus V_1$, and sometimes $V = V_1 \oplus V_0$, but we always mean the same thing.

Given the $\mathbb{Z}_2$-graded vector space V, consider the $\mathbb{K}$-linear transformations $\mathrm{End}(V)$ of $V = V_1 \oplus V_0$. The elements $X \in \mathrm{End}(V_1 \oplus V_0)$ decompose into blocks:

$$X = \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & \mathsf{D} \end{pmatrix} , \tag{1.15}$$

where $\mathsf{A} \in \mathrm{End}(V_1)$, $\mathsf{B} \in \mathrm{Hom}(V_0, V_1)$, $\mathsf{C} \in \mathrm{Hom}(V_1, V_0)$, and $\mathsf{D} \in \mathrm{End}(V_0)$. The vector space $\mathrm{End}(V)$ inherits from V a natural $\mathbb{Z}_2$-grading $\mathrm{End}(V) = \mathrm{End}(V)_0 \oplus \mathrm{End}(V)_1$ by

$$\begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & \mathsf{D} \end{pmatrix} = \begin{pmatrix} \mathsf{A} & 0 \\ 0 & \mathsf{D} \end{pmatrix} + \begin{pmatrix} 0 & \mathsf{B} \\ \mathsf{C} & 0 \end{pmatrix} , \tag{1.16}$$

i.e., $\mathrm{End}(V)_0 \simeq \mathrm{End}(V_0) \oplus \mathrm{End}(V_1)$ and $\mathrm{End}(V)_1 \simeq \mathrm{Hom}(V_0, V_1) \oplus \mathrm{Hom}(V_1, V_0)$. The *supertrace* on V is the linear function $\mathrm{STr}_V : \mathrm{End}(V) \to \mathbb{K}$ defined by

$$\mathrm{STr}_V\, X = \mathrm{Tr} X|_{V_0 \to V_0} - \mathrm{Tr} X|_{V_1 \to V_1} = \mathrm{Tr}\,\mathsf{D} - \mathrm{Tr}\,\mathsf{A} . \tag{1.17}$$

It has cyclic invariance in the $\mathbb{Z}_2$-graded sense: $\mathrm{STr}\,(XY) = (-1)^{|X|\,|Y|}\mathrm{STr}\,(YX)$.

As a $\mathbb{Z}_2$-graded vector space, $\mathrm{End}(V)$ carries a natural bracket operation,

$$[\,,\,] : \quad \mathrm{End}(V) \times \mathrm{End}(V) \to \mathrm{End}(V) ,$$

which for any two homogeneous elements $X, Y \in \mathrm{End}(V)$ is defined by

$$[X, Y] = XY - (-1)^{|X|\,|Y|} YX . \tag{1.18}$$

Thus the bracket of two odd elements is the anti-commutator, while in the other cases it is the commutator. This definition is extended to all of $\mathrm{End}(V)$ by linearity. The algebra $\mathrm{End}(V)$ equipped with the bracket $[\,,\,]$ is denoted by $\mathfrak{gl}(V)$, and is an example of a *Lie superalgebra*. If $V = \mathbb{K}^{p|q}$, one writes $\mathfrak{gl}(V) \equiv \mathfrak{gl}_{p|q}(\mathbb{K})$.

More generally a Lie superalgebra $\mathfrak{g}$ is a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ over $\mathbb{K}$ equipped with a bilinear bracket operation that has the following three properties:

$$\begin{aligned} &\text{(i)} \quad |[X,Y]| = |X| + |Y| \pmod 2 , \\ &\text{(ii)} \quad [X,Y] = -(-1)^{|X|\,|Y|}[Y,X] , \\ &\text{(iii)} \quad [X,[Y,Z]] = [[X,Y],Z] + (-1)^{|X|\,|Y|}[Y,[X,Z]] , \end{aligned}$$

for all $X,Y,Z \in \mathfrak{g}$ of definite parity. Property (iii) is called the super-Jacobi identity.

Example. — Let each of V_1 and V_0 carry a non-degenerate $\mathbb{K}$-bilinear form,

$$S:\ V_1 \times V_1 \to \mathbb{K} , \quad A:\ V_0 \times V_0 \to \mathbb{K} ,$$

and let S be symmetric and A alternating (which of course implies that the dimension of V_0 is even). On $V = V_1 \oplus V_0$ introduce a $\mathbb{K}$-bilinear form $Q : V \times V \to \mathbb{K}$ by the requirements that V_1 be Q-orthogonal to V_0, and that $Q|_{V_1 \times V_1} = S$ and $Q|_{V_0 \times V_0} = A$. A $\mathbb{Z}_2$-graded vector space V with such a form Q is called *orthosymplectic*.

Consider then the set of $\mathbb{K}$-linear transformations of V that are skew with respect $Q = S + A$. This set is a vector space, commonly denoted by $\mathfrak{osp}(V)$, which inherits from $\mathrm{End}(V)$ the $\mathbb{Z}_2$-grading (1.16), and skewness of a homogeneous element $X \in \mathrm{End}(V)$ here means that the relation

$$Q(Xv,v') + (-1)^{|X|\,|v|}Q(v,Xv') = 0$$

holds for all $v, v' \in V$ with homogeneous v.

It is not difficult to check that $\mathfrak{osp}(V)$ closes under the bracket operation (1.18) in $\mathfrak{gl}(V)$ and thus is a Lie superalgebra; one calls it the *orthosymplectic* Lie superalgebra of V. If $V_1 = \mathbb{K}^p$ and $V_0 = \mathbb{K}^q$ (with $q \in 2\mathbb{N}$), one writes $\mathfrak{osp}(V) = \mathfrak{osp}_{p|q}(\mathbb{K})$.

Returning to the general case, note that the supertrace of any bracket $[X,Y]$ in $\mathfrak{gl}(V)$ or $\mathfrak{osp}(V)$ vanishes. Indeed, by $(-1)^{|X|\,|Y|}\mathrm{STr}\,(YX) = \mathrm{STr}\,(XY)$ one has

$$\mathrm{STr}\,[X,Y] = \mathrm{STr}\left(XY - (-1)^{|X|\,|Y|}YX\right) = -\mathrm{STr}\,[X,Y] = 0 .$$

Besides the Lie superalgebra $\mathfrak{gl}(V)$, the $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$ carries another canonical structure: this is the graded-commutative algebra of maps

$$f : V_0 \to \wedge(V_1^*) ,$$

or functions on V_0 with values in the exterior algebra of the dual vector space of V_1. We want such maps to be holomorphic for $\mathbb{K} = \mathbb{C}$ and real-analytic for $\mathbb{K} = \mathbb{R}$. In the latter case, if $x^1, \ldots, x^p$ is a coordinate system for V_0, and $\xi^1, \ldots, \xi^q$ is a system of

linear coordinates for V_1 (and hence a system of generators of $\wedge V_1^*$), such a mapping is written in the form

$$f = \sum_{k=0}^{q} \sum_{i_1 < \ldots < i_k} f_{i_1 \ldots i_k}(x^1, \ldots, x^p)\, \xi^{i_1} \wedge \cdots \wedge \xi^{i_k} .$$

The wedge symbol $\wedge$ means exterior multiplication and will usually be omitted when its presence is clear from the context. A map $f : V_0 \to \wedge(V_1^*)$ is commonly referred to as a *superfunction on* V for short.

Consider now the related situation where the $\mathbb{Z}_2$-graded vector space is not V but $\mathrm{End}(V)$. Then there exists a distinguished superfunction, SDet, called the *superdeterminant*. To define it, let B be the tautological mapping (i.e., the identity map)

$$B : \ \mathrm{Hom}(V_0, V_1)^* \to \mathrm{Hom}(V_0, V_1)^* .$$

If $\{\mathsf{F}_i\}$ is a basis of $\mathrm{Hom}(V_0, V_1)$ with dual basis $\{\varphi^i\}$, this is expressed by

$$B = \sum \varphi^i \otimes \mathsf{F}_i .$$

Of course B does not depend on the choice of basis but is invariantly defined.

We now re-interpret B in the following way. The basis vectors $\mathsf{F}_i \in \mathrm{Hom}(V_0, V_1)$ are still to be viewed as linear maps from V_0 to V_1, and their destiny is to be composed with other maps; e.g., if $A \in \mathrm{GL}(V_1)$ and $D \in \mathrm{GL}(V_0)$, then $A^{-1}\mathsf{F}_i D^{-1} \in \mathrm{Hom}(V_0, V_1)$ is a meaningful expression. The dual basis vectors φ^i, on the other hand, will be regarded as coordinates for $\mathrm{Hom}(V_0, V_1)$; as such they are to be multiplied in the exterior sense. Thus we view them as a set of generators of the exterior algebra $\wedge \mathrm{Hom}(V_0, V_1)^*$.

Similarly, if $\{\tilde{\mathsf{F}}_j\}$ is a basis of $\mathrm{Hom}(V_1, V_0)$ with dual basis $\{\tilde\varphi^j\}$, let the tautological map $C : \ \mathrm{Hom}(V_1, V_0)^* \to \mathrm{Hom}(V_1, V_0)^*$ be expressed by

$$C = \sum \tilde\varphi^j \otimes \tilde{\mathsf{F}}_j ,$$

and let C be given the same re-interpretation as B.

In defining the superdeterminant, we assume the following associative product:

$$(\varphi^i \otimes \mathsf{F}_i)(\tilde\varphi^j \otimes \tilde{\mathsf{F}}_j) := \varphi^i \tilde\varphi^j \otimes \mathsf{F}_i \tilde{\mathsf{F}}_j . \tag{1.19}$$

(Technically speaking, this corresponds to a Grassmann envelope of the first kind [**2**].)

***Definition 1.5* (superdeterminant)**. — *If* $V = V_0 \oplus V_1$ *is a* $\mathbb{Z}_2$*-graded vector space,* $\mathrm{End}(V)$ *is understood to carry the canonical* $\mathbb{Z}_2$*-grading given by*

$$\begin{aligned} \mathrm{End}(V)_0 &\simeq \mathrm{End}(V_0) \oplus \mathrm{End}(V_1) , \\ \mathrm{End}(V)_1 &\simeq \mathrm{Hom}(V_0, V_1) \oplus \mathrm{Hom}(V_1, V_0) . \end{aligned}$$

Let $G = (\mathrm{GL}(V_1) \times \mathrm{GL}(V_0) \hookrightarrow \mathrm{End}(V)_0)$ *be the group of invertibles in* $\mathrm{End}(V)_0$. *Then the superdeterminant on* V *is the superfunction*

$$\begin{aligned} \mathrm{SDet}_V : \quad G &\to \wedge \mathrm{End}(V)_1^* , \\ A, D &\mapsto \frac{\mathrm{Det}(D)}{\mathrm{Det}(A)} \mathrm{Det}\Big(\mathrm{Id}_{V_1} - \textstyle\sum_{i,j} \varphi^i \tilde\varphi^j \otimes A^{-1}\mathsf{F}_i D^{-1} \tilde{\mathsf{F}}_j\Big)^{-1} , \end{aligned}$$

where the last factor is defined by expansion of the determinant using $\mathrm{Det} = \mathrm{e}^{\mathrm{Tr} \circ \ln}$,

$$\mathrm{Det}\Big(\mathrm{Id}_{V_1} - \textstyle\sum_{i,j} \varphi^i \tilde{\varphi}^j \otimes A^{-1} \mathsf{F}_i D^{-1} \tilde{\mathsf{F}}_j\Big)^{-1} = 1 + \textstyle\sum_{i,j} \varphi^i \tilde{\varphi}^j \, \mathrm{Tr}_{V_1} (A^{-1} \mathsf{F}_i D^{-1} \tilde{\mathsf{F}}_j) + \dots \,.$$

Remark. — The expansion terminates because of the nilpotency of the φ^i and $\tilde{\varphi}^j$.

It is convenient and standard practice in superanalysis to take the identifications $\sum \varphi^i \otimes \mathsf{F}_i \equiv B$ and $\sum \tilde{\varphi}^j \otimes \tilde{\mathsf{F}}_j \equiv C$ for granted and, with the blocks A, B, C, D assembled to a *supermatrix* Ξ as in (1.15), adopt for the superdeterminant function $\mathrm{SDet}_V : G \to \wedge \mathrm{End}(V)_1^*$ the short-hand notation $\mathrm{SDet}(\Xi)$.

Proposition 1.6. — *The superdeterminant has the following equivalent expressions:*

$$\mathrm{SDet}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{\mathrm{Det}(D)}{\mathrm{Det}(A - BD^{-1}C)} = \frac{\mathrm{Det}(D - CA^{-1}B)}{\mathrm{Det}(A)} \,.$$

Proof. — The first expression results from Def. 1.5 on rewriting $\sum \varphi^i \otimes \mathsf{F}_i = B$ and $\sum \tilde{\varphi}^j \otimes \tilde{\mathsf{F}}_j = C$ and using the multiplicativity of the determinant. The second expression follows from the first one with the help of the relation

$$\mathrm{Det}(\mathrm{Id}_{V_1} - A^{-1}BD^{-1}C)^{-1} = \mathrm{Det}(\mathrm{Id}_{V_0} - D^{-1}CA^{-1}B) \,,$$

which in turn is a straightforward consequence of $\ln \circ \mathrm{Det} = \mathrm{Tr} \circ \ln$, the Taylor series of the logarithm $x \mapsto \ln(1-x)$, and the alternating property of the exterior product $\wedge$. □

Remark. — Note that the reciprocal of a superdeterminant,

$$\mathrm{SDet}^{-1}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{\mathrm{Det}(A - BD^{-1}C)}{\mathrm{Det}(D)} \,,$$

exists whenever the block D possesses an inverse.

1.3. The good object to consider. — Consider now the special situation of a $\mathbb{Z}_2$-graded complex vector space $V = V_1 \oplus V_0$ with tensor-product structure:

$$V = U \otimes \mathbb{C}^N \simeq (U_1 \otimes \mathbb{C}^N) \oplus (U_0 \otimes \mathbb{C}^N) \,,$$

where $U = U_1 \oplus U_0$ is another $\mathbb{Z}_2$-graded vector space, and $\mathbb{C}^N$ is a fundamental module for $K = \mathrm{U}_N$. If $k \in K$, and $\Xi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a supermatrix where $B = \sum \varphi^i \otimes F_i$ and $C = \sum \tilde{\varphi}^j \otimes \tilde{F}_j$ as above, the superdeterminant of $\mathrm{Id}_V - \Xi \otimes k$ is defined in the obvious manner, i.e., by the natural embedding $\mathrm{End}(U) \otimes \mathrm{End}(\mathbb{C}^N) \hookrightarrow \mathrm{End}(U \otimes \mathbb{C}^N)$.

For present purposes, let U_0 and U_1 be Hermitian vector spaces of equal dimension:

$$U_1 = \mathbb{C}^n \,, \quad U_0 = \mathbb{C}^n \,.$$

Definition 1.7. — *If* $A = \Xi|_{U_1 \to U_1}$ *and* $D = \Xi|_{U_0 \to U_0}$ *let* $\mathscr{D} \subset \mathrm{End}(U)_0$ *be the subset where* A *and* D *are invertible and the spectrum of* D *avoids the unit circle. The key*

object of our approach for the case of $K = \mathrm{U}_N$ is the superfunction $\chi : \mathscr{D} \to \wedge \mathrm{End}(U)_1^*$ defined as the integral

$$\chi(\Xi) = \int_K \mathrm{SDet}(\mathrm{Id}_V - \Xi \otimes k)^{-1} \, dk \,. \tag{1.20}$$

Remark. — By the spectral condition on D, the linear operator $\mathrm{Id}_{V_0} - D \otimes k$ has an inverse for all $k \in \mathrm{U}_N$, and this by the previous remark ensures that (1.20) exists.

To appreciate why the superfunction χ of (1.20) plays such an important role, evaluate its $\mathbb{C}$-number part on the diagonal matrices $t \in \mathrm{End}(U)$ of (1.2), which gives

$$\chi(t) = \int\limits_{\mathrm{U}_N} \prod_{k=1}^{n} \frac{\mathrm{Det}(\mathrm{Id}_N - \mathrm{e}^{\mathrm{i}\psi_k} u)}{\mathrm{Det}(\mathrm{Id}_N - \mathrm{e}^{\phi_k} u)} \, du \,.$$

Comparison with the definitions (1.4) and (1.5) shows that this is exactly the autocorrelation function we want to compute.

On the other hand, from Howe's theory of dual pairs the superfunction χ of Def. 1.7 can be interpreted as a primitive character of the Lie supergroup $\mathfrak{G} = \mathrm{GL}_{n|n}$. The task of conveying the precise meaning of this message will occupy us for much of the present paper. The essential points are summarized in the following subsection.

Please take note of the following strategic aspect. We want to prove Thm. 1.1, which is a statement about the complex-valued function $t \mapsto \chi(t)$, the autocorrelation function of ratios of characteristic polynomials defined in (1.5). We will therefore be aiming not at the superfunction $\chi(\Xi)$, but rather at the ordinary function $\chi(t)$. Our route to $\chi(t)$, however, does lead through the extended object $\chi(\Xi)$. Indeed, it is the very existence of $\chi(\Xi)$ that will put us in a position to deduce various strong properties of $\chi(t)$ and thereby establish the expression asserted by Thm. 1.1 in the full range $N \in \mathbb{N}$.

1.4. Outline of strategy. — The superfunction $\chi(\Xi)$ defined by (1.20) is analytic, but only piecewise so, because of the singularities that occur when one or several of the eigenvalues of $D = \Xi|_{\mathrm{End}(U_0)}$ hit the unit circle. Since we want a correlation of signature (p,q), i.e., of a product of p ratios involving $u \in \mathrm{U}_N$ and q ratios involving the complex conjugate $\bar{u}$, we are interested in the domain of analyticity, $\mathscr{D}_{p,q}$, where p eigenvalues e^{ϕ_j} of $D \in \mathrm{GL}(\mathbb{C}^{p+q})$ lie inside the unit circle ($\mathfrak{Re}\,\phi_j < 0$), and q eigenvalues e^{ϕ_l} lie outside ($\mathfrak{Re}\,\phi_l > 0$).

The main step of our approach is to establish a certain system of differential equations obeyed by $\chi(\Xi)$ and hence by $\chi(t)$. In this endeavor we have to handle the unbounded operators of a non-compact group acting on an infinite-dimensional space. For that reason, most of the discussion of $\chi(\Xi)$ will be carried out not in the complex space $\mathscr{D}_{p,q}$ but in a *real-analytic domain* $M \subset \mathscr{D}_{p,q}$, which is specified as follows.

Let $T = T_1 \times T_0 \subset \mathrm{GL}(U_1) \times \mathrm{GL}(U_0)$ be the Abelian semigroup with factors

$$T_1 = \mathrm{U}_1^{p+q} \,, \quad T_0 = (0,1)^p \times (1,\infty)^q \,,$$

where $\mathrm{U}_1 \equiv \mathrm{S}^1$ is the unit circle in the complex number field $\mathbb{C}$. We identify this "torus" T with the set of diagonal matrices $t = (t_1, t_0)$ of (1.2),

$$t_1 = \mathrm{diag}\,(\mathrm{e}^{\mathrm{i}\psi_1}, \ldots, \mathrm{e}^{\mathrm{i}\psi_n}) \;, \quad t_0 = \mathrm{diag}\,(\mathrm{e}^{\phi_1}, \ldots, \mathrm{e}^{\phi_n}) \quad (n = p+q) \;,$$

where all of the variables $\psi_1, \ldots, \psi_n, \phi_1, \ldots, \phi_n$ are now *real-valued*, and the ϕ_k are restricted to the range

$$\phi_j < 0 < \phi_l \quad (1 \le j \le p < l \le n).$$

Our real-analytic domain M will be a direct product $M = M_1 \times M_0$. To describe the second space M_0 let $\mathrm{U}_{p,q}$ be the pseudo-unitary group determined by the signature operator $s = \mathrm{diag}(\mathrm{Id}_p, -\mathrm{Id}_q)$ on the Hermitian vector space $U_0 \equiv \mathbb{C}^n = \mathbb{C}^p \oplus \mathbb{C}^q$. Letting $g \cdot t_0 = g t_0 g^{-1}$ denote the adjoint action of $g \in \mathrm{U}_{p,q}$ on $T_0 \subset \mathrm{GL}(\mathbb{C}^n)$, we define

$$M_0 := \mathrm{U}_{p,q} \cdot T_0 \;,$$

i.e., M_0 is the union of all adjoint orbits of $\mathrm{U}_{p,q}$ on the points of T_0. Similarly, let

$$M_1 := \mathrm{U}_{p+q} \cdot T_1 = \mathrm{U}_{p+q} \;,$$

the union of adjoint orbits of the unitary group $\mathrm{U}(U_1) = \mathrm{U}_{p+q}$ on T_1, which is the same as U_{p+q}. Note that $M_0 \subset \mathrm{GL}(U_0)$ is a real-analytic submanifold of dimension n^2, and so is $M_1 \subset \mathrm{GL}(U_1)$. We shall study the superfunction $\chi(\Xi)$ on the product manifold

$$M := M_1 \times M_0 \;,$$

i.e., for $\Xi|_{\mathrm{End}(U_1)} \in M_1$ and $\Xi|_{\mathrm{End}(U_0)} \in M_0$.

Now recall that $V = U \otimes \mathbb{C}^N = V_0 \oplus V_1$, and regarding this as $V = \mathbb{C}^{n|n} \otimes \mathbb{C}^N$, let a Howe dual pair $(\mathfrak{gl}_{n|n}, \mathrm{U}_N)$ [**12**] act on V as its fundamental module. In Sect. 2 it is shown that the function $Z: \; T \times \mathrm{U}_N \to \mathbb{C}$ defined by

$$Z(t,u) = \mathrm{SDet}(\mathrm{Id}_V - t \otimes u)^{-1}$$

can be regarded as the character of that pair acting on a spinor-oscillator module $\mathscr{A}_V$ of V, i.e., the tensor product of an exterior algebra (built from V_1) with a symmetric algebra (from V_0). In physics, $\mathscr{A}_V$ is called a Fock space for bosons and fermions. Since the action of the Howe pair $(\mathfrak{gl}_{n|n}, \mathrm{U}_N)$ on $\mathscr{A}_V$ is known to be multiplicity-free, the superfunction $\chi(\Xi)$ given by (1.20) is the character of an *irreducible* $\mathfrak{gl}_{n|n}$-representation.

For better readability, we first establish this statement in Sect. 2 for the easy case of a diagonal matrix $t \in T$. In Sect. 4 we then introduce a certain real-analytic supermanifold $\mathscr{F}$ inside the complex supergroup $\mathrm{GL}_{n|n}$ carrying left and right actions of the Lie superalgebra $\mathfrak{gl}_{n|n}$, and we explain how to make sense of the character $\chi(\Xi)$ as a section of $\mathscr{F}$ over the product of real-analytic domains $M = M_1 \times M_0$.

Although the character χ is atypical, and is not covered by the Weyl character formula and its known extensions, it can nonetheless be computed by a method developed in this paper. In Sect. 4 we accumulate a wealth of information about it:

- The function $\chi: \; T \to \mathbb{C}$ is analytic and W-invariant.
- The character χ is a linear combination of exponential functions $\mathrm{e}^{\sum_k (\mathrm{i} m_k \psi_k - n_k \phi_k)}$ where the weights are in the range $n_j \le 0 \le m_k \le N \le n_l$ for $1 \le j \le p < l \le n$.

- $\chi(\Xi)$ is degenerate with the character of the trivial representation; thus $\chi(t)$ is annihilated by the radial parts of all of the Laplace-Casimir operators of $\mathfrak{gl}_{n|n}$.

Using the radial parts given by Berezin [**2**], we show that the problem posed by properties (A-C) admits one and only one solution: that stated in Thm. 1.1. Please note that although the proof is carried out for real variables ψ_k and ϕ_k, the final statement immediately extends to complex values of these variables by analytic continuation.

As a historical note, let us mention that our treatment relies on just two sources: Berezin [**2**], and Howe [**12**] – and these have existed in mathematics for 30 years now. Indeed, Berezin's results are from 1975 and Howe's paper was written in 1976.

2. Autocorrelation function of ratios as a primitive character

2.1. Clifford algebra, spinor module, and character formula. — Starting from a complex vector space $V = \mathbb{C}^d$, denote the dual vector space by V^* and form the direct sum $W := V \oplus V^*$. On W define a symmetric bilinear form $S : W \times W \to \mathbb{C}$ by

$$S(v+\varphi, v'+\varphi') = \varphi(v') + \varphi'(v) \,. \tag{2.1}$$

Then let $\mathfrak{c}(W)$ be the Clifford algebra of W, i.e., the associative algebra generated by $W \oplus \mathbb{C}$ with anti-commutation relations

$$ww' + w'w = S(w, w') \,. \tag{2.2}$$

$\mathfrak{c}(W)$ is $\mathbb{Z}_2$-graded as a vector space by the dichotomy of the degree being either even or odd. Via the relations (2.2) the Clifford algebra $\mathfrak{c}(W)$ carries the natural structure of a Lie superalgebra, with the fundamental bracket being $[w,w'] := ww' + w'w$.

2.1.1. Spinor representation. — Basic to our approach is the spinor representation of $\mathfrak{c}(W)$. Given the polarization $W = V \oplus V^*$ there exist *two* natural realizations of it. Both of them are used in the quantum field theory of Dirac fermions, where the positive and negative parts of the Dirac operator call for two different quantization schemes.

Here, too, we need both realizations. To define the first one, consider the exterior algebra $\wedge(V) = \oplus_{k=0}^{d} \wedge^k (V)$. On it one has the operations of exterior multiplication ε,

$$V \times \wedge^k(V) \to \wedge^{k+1}(V) \,, \quad (v, a) \mapsto \varepsilon(v)\, a = v \wedge a \,,$$

and the operation of taking the inner product (or contraction) ι,

$$V^* \times \wedge^k(V) \to \wedge^{k-1}(V) \,, \quad (\varphi, a) \mapsto \iota(\varphi)\, a \,,$$

where the operator $\iota(\varphi)$ is the anti-derivation of $\wedge(V)$ defined (for $v, v' \in V$) by

$$\iota(\varphi)\, 1 = 0 \,, \quad \iota(\varphi)\, v = \varphi(v) \,, \quad \iota(\varphi)\, (v \wedge v') = \varphi(v)\, v' - \varphi(v')\, v \,,$$

and so on. The algebraic relations obeyed by these operators of exterior and interior multiplication are the so-called *canonical anti-commutation relations* (CAR):

$$\begin{aligned} &\varepsilon(v)\, \varepsilon(v') + \varepsilon(v')\, \varepsilon(v) = 0 \,, \quad \iota(\varphi)\, \iota(\varphi') + \iota(\varphi')\, \iota(\varphi) = 0 \,, \\ &\iota(\varphi)\, \varepsilon(v) + \varepsilon(v)\, \iota(\varphi) = \varphi(v)\, \mathrm{Id}_{\wedge(V)} \,. \end{aligned} \tag{2.3}$$

If one assigns to $w = v + \varphi \in V \oplus V^* = W$ a linear operator on $\wedge(V)$ by

$$v + \varphi \mapsto \varepsilon(v) + \iota(\varphi) , \tag{2.4}$$

then this assignment preserves the Clifford algebra relations (2.2) by virtue of the relations (2.3). Therefore, (2.4) defines a representation of the Clifford algebra $\mathfrak{c}(W)$. It is called the spinor representation of $\mathfrak{c}(W)$ on the spinor module $\wedge(V)$.

Next, consider realizing the same object $v + \varphi$ as an operator on the exterior algebra $\wedge(V^*)$ of the dual vector space V^*. This is done by the assignment

$$W = V \oplus V^* \ni v + \varphi \mapsto \iota(v) + \varepsilon(\varphi) , \tag{2.5}$$

where $\iota(v) : \wedge^k(V^*) \mapsto \wedge^{k-1}(V^*)$ and $\varepsilon(\varphi) : \wedge^k(V^*) \mapsto \wedge^{k+1}(V^*)$ now mean contraction by the vector v and exterior multiplication by the linear form φ. These operations still satisfy the canonical anti-commutation relations and hence give a second representation of the Clifford algebra $\mathfrak{c}(W)$.

The second representation is isomorphic to the first one. Indeed, if we fix a generator $\Omega \in \wedge^d(V^*)$ then the mapping $\tau : \wedge^k(V) \to \wedge^{d-k}(V^*)$ defined for $k = 0, 1, \ldots, d$ by

$$\wedge^0(V) \ni 1 \mapsto \Omega \in \wedge^d(V^*) , \quad \wedge^1(V) \ni \varepsilon(v) \cdot 1 \mapsto \iota(v)\Omega \in \wedge^{d-1}(V^*) ,$$

and so on, sends $\varepsilon(v)$ to $\iota(v)$ and $\iota(\varphi)$ to $\varepsilon(\varphi)$. It is called a *particle-hole transformation* in physics. As a small remark, note that τ is an isomorphism of $\mathbb{Z}_2$-graded vector spaces only if d is even. (When d is odd, τ exchanges the even and odd subspaces.)

2.1.2. $\mathrm{GL}(V)$ *character formula.* — Let $\mathrm{GL}(V)$ be the group of invertible complex linear transformations of V. Denoting the action of $\mathrm{GL}(V)$ on its fundamental module V as $v \mapsto gv$, one has an induced representation $\sigma^k : \mathrm{GL}(V) \to \mathrm{GL}(\wedge^k V)$ on $\wedge^k(V)$ by

$$\sigma^k(g)(v_1 \wedge v_2 \wedge \cdots \wedge v_k) = gv_1 \wedge gv_2 \wedge \cdots \wedge gv_k .$$

By taking the direct sum of these representations for $k = 0, 1, \ldots, d$, one gets a canonical $\mathrm{GL}(V)$-representation σ on the exterior algebra $\wedge(V)$:

$$\sigma : \ \mathrm{GL}(V) \to \mathrm{GL}(\wedge V) . \tag{2.6}$$

It is a standard fact of multi-linear algebra that the alternating sum of characters of the representations σ^k is generated by a determinant:

$$\textstyle\sum_{k=0}^{d} (-t)^k \,\mathrm{Tr}\, \sigma^k(g) = \mathrm{Det}(\mathrm{Id}_V - tg) \qquad (t \in \mathbb{C}) .$$

Setting $t = 1$ and denoting by $\mathrm{STr}_{\wedge V}$ the operation of taking the supertrace over the $\mathbb{Z}_2$-graded vector space $\wedge(V) = \wedge^{\mathrm{even}}(V) \oplus \wedge^{\mathrm{odd}}(V)$, we write this formula as

$$\mathrm{STr}_{\wedge V}\, \sigma(g) = \mathrm{Det}(\mathrm{Id}_V - g) . \tag{2.7}$$

If we specialize to the tensor-product situation $V = \mathbb{C}^p \otimes \mathbb{C}^N$ and $g = t \otimes u$, with $u \in \mathrm{U}(\mathbb{C}^N)$ and $t \in \mathrm{GL}(\mathbb{C}^p)$ a diagonal transformation $t = \mathrm{diag}\,(t_1, \ldots, t_p)$, then we obtain

$$\mathrm{STr}_{\wedge V}\, \sigma(t \otimes u) = \textstyle\prod_{j=1}^{p} \mathrm{Det}(\mathrm{Id}_N - t_j\, u) ,$$

which is ”half” of the numerator in our basic product of ratios (1.1).

In order to produce the other half, which involves the complex conjugates $\bar{u}$, we are going to employ a $\mathrm{GL}(V)$-representation on $\wedge(V^*)$. Observing that $g \in \mathrm{GL}(V)$ acts on

the dual vector space V^* by $g\varphi := \varphi \circ g^{-1}$, we take $\widetilde{\sigma} : \mathrm{GL}(V) \to \mathrm{GL}(\wedge V^*)$ to be the direct sum $\widetilde{\sigma} = \oplus_{k=0}^{d} \widetilde{\sigma}^k$ of representations $\widetilde{\sigma}^k : \mathrm{GL}(V) \to \mathrm{GL}(\wedge^k V^*)$ defined by

$$\widetilde{\sigma}^k(g)(\varphi_1 \wedge \cdots \wedge \varphi_k) = \mathrm{Det}(g)\,(\varphi_1 \circ g^{-1}) \wedge \cdots \wedge (\varphi_k \circ g^{-1})\;. \tag{2.8}$$

Notice that $\widetilde{\sigma}^d : \mathrm{GL}(V) \to \mathrm{GL}(\wedge^d V^*)$ is the trivial representation, $\widetilde{\sigma}^d(g)\Omega = \Omega$, isomorphic to $\sigma^0 : \mathrm{GL}(V) \to \mathrm{GL}(\wedge^0 V)$. As a matter of fact, the insertion of the determinant multiplier makes the two representations σ and $\widetilde{\sigma}$ isomorphic by the particle-hole transformation $\tau : \wedge^k(V) \to \wedge^{d-k}(V^*)$. We will see this at the infinitesimal level below. Since τ for $d \in 2\mathbb{N}-1$ interchanges the even and odd subspaces we have

$$\mathrm{STr}_{\wedge V^*}\widetilde{\sigma}(g) = (-1)^d\,\mathrm{Det}(\mathrm{Id}_V - g) = \mathrm{Det}(g - \mathrm{Id}_V)\;. \tag{2.9}$$

How are σ and $\widetilde{\sigma}$ related to the spinor representation (2.4) of $\mathfrak{c}(W)$? To answer that question, let $\{e_1, \ldots, e_d\}$ be a basis of V with corresponding dual basis $\{f^1, \ldots, f^d\}$ of V^*, and consider the sequence of linear mappings

$$\mathfrak{gl}(V) \to V \otimes V^* \to \mathfrak{gl}(\wedge V)\;, \quad X \mapsto \sum (Xe_i) \otimes f^i \mapsto \sum \varepsilon(Xe_i)\iota(f^i)\;.$$

The first map is a Lie algebra isomorphism. The second map turns $v \otimes \varphi \in V \otimes V^*$ into an element $v\varphi$ of the Clifford algebra $\mathfrak{c}(W)$ by dropping the tensor product and then sends $v\varphi$ to an operator on $\wedge(V)$ by (2.4). By the CAR relations (2.3), this map $v \otimes \varphi \mapsto \varepsilon(v)\iota(\varphi)$ is a Lie algebra homomorphism. Altogether, the composite map $X \mapsto \sum \varepsilon(Xe_i)\iota(f^i)$ is a representation of $\mathfrak{gl}(V)$ on $\wedge(V)$. By the same token, the linear mapping $X \mapsto \sum \iota(Xe_i)\varepsilon(f^i)$ is a representation of $\mathfrak{gl}(V)$ on $\wedge(V^*)$. These Lie algebra representations are related to the Lie group representations σ and $\widetilde{\sigma}$ as follows.

Proposition 2.1. — *Taking the differential of the representations* $\sigma : \mathrm{GL}(V) \to \mathrm{GL}(\wedge V)$ *and* $\widetilde{\sigma} : \mathrm{GL}(V) \to \mathrm{GL}(\wedge V^*)$ *at the neutral element one has*

$$\sigma_*(X) := \frac{d}{dt}\sigma(\mathrm{e}^{tX})\Big|_{t=0} = \sum_{i=1}^{d} \varepsilon(Xe_i)\iota(f^i)\;, \qquad \widetilde{\sigma}_*(X) = \sum_{i=1}^{d} \iota(Xe_i)\varepsilon(f^i)\;.$$

Proof. — Apply the operator $\sigma'(X) := \sum \varepsilon(Xe_i)\iota(f^i)$ to any element $v \in V$:

$$\sigma'(X)v = \textstyle\sum_i \varepsilon(Xe_i) f^i(v) = \varepsilon(Xv) \cdot 1 = Xv = \frac{d}{dt}\Big|_{t=0} \sigma^1(\mathrm{e}^{tX})v = \sigma_*(X)v\;.$$

Thus $\sigma'(X) = \sigma_*(X)$ on V. Now the operator $D \equiv \sigma'(X)$ satisfies the Leibniz rule

$$D(v_1 \wedge v_2) = Dv_1 \wedge v_2 + v_1 \wedge Dv_2\;,$$

as follows easily from the definition of exterior and interior multiplication. Thus $\sigma'(X)$ is a derivation of the exterior algebra $\wedge(V)$ just as the differential $\sigma_*(X)$ is. Therefore, since $\sigma'(X)$ and $\sigma_*(X)$ agree on the generating vector space V, they agree on $\wedge(V)$.

In the case of $\widetilde{\sigma}'(X) := \sum \iota(Xe_i)\varepsilon(f^i)$ one uses CAR to write $\widetilde{\sigma}'(X) = \mathrm{Tr}(X)\mathrm{Id}_{\wedge V^*} + \sum \varepsilon(-X^{\mathrm{t}} f^i)\iota(e_i)$. The first term arises from linearizing $\mathrm{e}^{tX} \mapsto \mathrm{Det}(\mathrm{e}^{tX})$ at $t = 0$. The second term is a derivation of $\wedge(V^*)$ and the rest of the argument goes as above. ☐

What we have told up to now was the ”fermionic” variant (in physics language) of a story that has to be told twice.

2.2. Weyl algebra and oscillator module. — For the second variant of our story – the "bosonic" one – consider again $V=\mathbb{C}^d$ and $W=V\oplus V^*$, but now equipped with the *alternating* bilinear form $A: W\times W\to\mathbb{C}$ given by

$$A(v+\varphi,v'+\varphi')=\varphi(v')-\varphi'(v)\,. \tag{2.10}$$

The associative algebra generated by $W\oplus\mathbb{C}$ with the relations

$$ww'-w'w=A(w,w') \tag{2.11}$$

is denoted by $\mathfrak{w}(W)$, and is called the Weyl algebra of W.

To construct a representation of $\mathfrak{w}(W)$, let $\mathrm{S}(V)=\oplus_{k\geq 0}\mathrm{S}^k(V)$ be the symmetric algebra of V, and consider on it the operations of multiplication μ and derivation δ:

$$\begin{aligned} &V\times\mathrm{S}^k(V)\to\mathrm{S}^{k+1}(V)\,,\quad (v,a)\mapsto\mu(v)\,a=va=av\,,\\ &V^*\times\mathrm{S}^k(V)\to\mathrm{S}^{k-1}(V)\,,\quad (\varphi,a)\mapsto\delta(\varphi)\,a\,.\end{aligned}$$

The operator $\delta(\varphi)$ obeys the Leibniz rule $\delta(\varphi)\,v^k=k\,v^{k-1}\varphi(v)$. The algebraic relations satisfied by μ and δ are the *canonical commutation relations* (CCR):

$$\begin{aligned} &\mu(v)\,\mu(v')-\mu(v')\,\mu(v)=0\,,\quad \delta(\varphi)\,\delta(\varphi')-\delta(\varphi')\,\delta(\varphi)=0\,,\\ &\delta(\varphi)\,\mu(v)-\mu(v)\,\delta(\varphi)=\varphi(v)\,\mathrm{Id}_{\mathrm{S}V}\,.\end{aligned} \tag{2.12}$$

Letting $w=v+\varphi\in V\oplus V^*=W$ act on $\mathrm{S}(V)$ by

$$v+\varphi\mapsto\mu(v)+\delta(\varphi)\,, \tag{2.13}$$

one gets a representation of the Weyl algebra $\mathfrak{w}(W)$ on $\mathrm{S}(V)$. The representation space $\mathrm{S}(V)$ is sometimes referred to as the *oscillator module* of $\mathfrak{w}(V\oplus V^*)$, as it carries the so-called oscillator representation of the metaplectic group of W.

A Hermitian structure $\langle\,,\rangle$ on V induces a canonical Hermitian structure on $\mathrm{S}(V)$ as follows. Let a complex anti-linear isomorphism $c: V\to V^*$ be defined by $c\,v=\langle v,\cdot\rangle$, and extend this to a mapping $c:\mathrm{S}(V)\to\mathrm{S}(V^*)$ in the natural manner, i.e., by applying the map c to every factor of an element of $\mathrm{S}(V)$. For example, $c(vv')=(cv)(cv')$. Now, using the canonical pairing of the vector space $\mathrm{S}(V)$ with its dual vector space $\mathrm{S}(V)^*=\mathrm{S}(V^*)$ one defines a Hermitian scalar product $\langle\,,\rangle_{\mathrm{S}V}:\mathrm{S}(V)\times\mathrm{S}(V)\to\mathbb{C}$ by

$$\langle\Phi,\Phi'\rangle_{\mathrm{S}V}=(c\,\Phi)(\Phi')\,.$$

With respect to this Hermitian structure of $\mathrm{S}(V)$ one has

$$\mu(v)^\dagger=\delta(c\,v)\,, \tag{2.14}$$

and in this sense the operator δ is adjoint to μ. (Throughout this paper the symbol $\dagger$ denotes the operation of taking the adjoint w.r.t. a given Hermitian structure.)

There exists a second way of implementing the relations (2.11) of the Weyl algebra $\mathfrak{w}(W)$. Instead of $\mathrm{S}(V)$ consider the symmetric algebra $\mathrm{S}(V^*)$ of the dual vector space V^*, and let $v+\varphi\in W$ operate on $\mathrm{S}(V^*)$ by

$$v+\varphi\mapsto\delta(v)-\mu(\varphi)\,. \tag{2.15}$$

With the modified sign ($\delta\to\mu$ and $\mu\to-\delta$), which is forced by the alternating nature of the basic form (2.10), this still defines a representation of $\mathfrak{w}(W)$.

A Hermitian structure of V still induces a canonical Hermitian structure of $\mathrm{S}(V^*)$, and the relations (2.14) continue to hold in the adapted form $\mu(\varphi)^\dagger = \delta(c^{-1}\varphi)$, or $\delta(v)^\dagger = \mu(cv)$. Our full setup below will utilize both representations (2.13) and (2.15).

2.3. Oscillator character. — Let the complex Lie group $\mathrm{GL}(V)$ still act on its fundamental module $V = \mathbb{C}^d$ as $v \mapsto gv$, but now consider the induced $\mathrm{GL}(V)$-representations ω^k on the symmetric powers $\mathrm{S}^k(V)$:

$$\omega^k(g)(v_1 \cdots v_k) = (gv_1) \cdots (gv_k) \ .$$

These combine to a representation ω on the symmetric algebra $\mathrm{S}(V)$:

$$\oplus_{k \geq 0}\, \omega^k = \omega : \ \mathrm{GL}(V) \to \mathrm{GL}(\mathrm{S}V) \ . \tag{2.16}$$

Since the sum over symmetric powers is an infinite sum, the character of this representation does not exist for all $g \in \mathrm{GL}(V)$. Rather, the following is true.

Equipping $V = \mathbb{C}^d$ with its standard Hermitian structure $\langle\, , \rangle$, let $g \mapsto g^\dagger$ denote the operation of taking the adjoint with respect to $\langle\, , \rangle$. Then consider the set

$$\mathrm{H}^<(V) = \{h \in \mathrm{GL}(V) \mid h^\dagger h < \mathrm{Id}_V\} \ , \tag{2.17}$$

i.e., the set of elements $h \in \mathrm{GL}(V)$ with the property $\langle hv, hv\rangle < \langle v, v\rangle$ for all $v \in V$. Clearly, every eigenvalue λ of $h \in \mathrm{H}^<(V)$ has absolute value $|\lambda| < 1$. If $g \in \mathrm{GL}(V)$ and $h \in \mathrm{H}^<(V)$, then $h^\dagger h < \mathrm{Id}_V$ implies $(hg)^\dagger hg < g^\dagger g$. Thus $\mathrm{H}^<(V)$ is a semigroup.

Now, if $g \in \mathrm{GL}(V)$ can be diagonalized and has eigenvalues $\lambda_1, \ldots, \lambda_d$, the character of the representation $\omega^k : \mathrm{GL}(V) \to \mathrm{GL}(\mathrm{S}^k V)$ takes on g the value

$$\mathrm{Tr}\, \omega^k(g) = \sum_{1 \leq i_1 \leq \ldots \leq i_k \leq d} \lambda_{i_1} \cdots \lambda_{i_k} \ .$$

If the additional requirement $g \in \mathrm{H}^<(V)$ is imposed, these values can be convergently summed over all symmetric powers $k \geq 0$ to give

$$\sum_{k=0}^{\infty} \mathrm{Tr}\, \omega^k(g) = \sum_{\{n_i\} \in (\mathbb{N} \cup \{0\})^d} \lambda_1^{n_1} \cdots \lambda_d^{n_d} = \prod_{i=1}^{d} (1 - \lambda_i)^{-1} \ .$$

Clearly, the right-hand side is the reciprocal of a determinant, $\mathrm{Det}^{-1}(\mathrm{Id}_V - g)$. Thus the character of the representation (2.16) can be written as

$$\mathrm{Tr}_{\mathrm{S}V}\, \omega(g) = \mathrm{Det}^{-1}(\mathrm{Id}_V - g) \tag{2.18}$$

for diagonalizable $g \in \mathrm{H}^<(V)$, in which form the formula extends to the case of general $g \in \mathrm{H}^<(V)$. Indeed, by the Jordan decomposition every g is a sum of semisimple (i.e., diagonalizable) and nilpotent parts, and the nilpotent part of g contributes neither to the character on the left-hand side nor to the determinant on the right-hand side.

Besides the $\mathrm{GL}(V)$-representation ω on $\mathrm{S}(V)$, we also need the representation $\widetilde{\omega}$: $\mathrm{GL}(V) \to \mathrm{GL}(\mathrm{S}V^*)$ which is defined as the direct sum $\widetilde{\omega} = \oplus_{k \geq 0}\, \widetilde{\omega}^k$,

$$\widetilde{\omega}^k(g)(\varphi_1 \cdots \varphi_k) = \mathrm{Det}^{-1}(g)(\varphi_1 \circ g^{-1}) \cdots (\varphi_k \circ g^{-1}) \ ,$$

of components $\widetilde{\omega}^k : \mathrm{GL}(V) \to \mathrm{GL}(\mathrm{S}^k V^*)$. An important observation is that although we have $\sigma = \tau^{-1}\widetilde{\sigma}\tau$ on the fermionic side, the representations ω and $\widetilde{\omega}$ are *not* isomorphic. (There is no such thing as a particle-hole transformation τ for bosons.) Indeed, while the character of ω exists on $\mathrm{H}^<(V)$, that of $\widetilde{\omega}$ exists on the opposite semigroup $\mathrm{H}^>(V) = \{h \in \mathrm{GL}(V) \mid h^\dagger h > \mathrm{Id}_V\}$, where it has the value

$$\mathrm{Tr}_{\mathrm{S}V^*}\, \widetilde{\omega}(h) = \mathrm{Det}^{-1}(h)\mathrm{Det}^{-1}(\mathrm{Id}_V - h^{-1}) = \mathrm{Det}^{-1}(h - \mathrm{Id}_V)\ . \tag{2.19}$$

The following statement is an analog of Prop. 2.1 and is proved in the same way.

Proposition 2.2. — *Taking the differential of the representations* $\omega : \mathrm{GL}(V) \to \mathrm{GL}(\mathrm{S}V)$ *and* $\widetilde{\omega} : \mathrm{GL}(V) \to \mathrm{GL}(\mathrm{S}V^*)$ *at the neutral element one has*

$$\omega_*(X) := \sum_{i=1}^{d} \mu(Xe_i)\delta(f^i)\ , \quad \widetilde{\omega}_*(X) = -\sum_{i=1}^{d} \delta(Xe_i)\mu(f^i)\ .$$

2.4. Enter supersymmetry. — Drawing on the material of Sects. 2.1–2.3, we can now express the product of ratios of characteristic polynomials (1.1) as a product of characters. Let V be a $\mathbb{Z}_2$-graded vector space which decomposes as

$$V = V_1 \oplus V_0 = (V_1^+ \oplus V_1^-) \oplus (V_0^+ \oplus V_0^-)\ , \tag{2.20}$$

and introduce the following tensor product of spinor and oscillator modules:

$$\mathscr{V} := \wedge(V_1^+) \otimes \wedge(V_1^{-\,*}) \otimes \mathrm{S}(V_0^+) \otimes \mathrm{S}(V_0^{-\,*})\ . \tag{2.21}$$

Let a product of representations

$$R :\ \mathbf{G} \equiv \mathrm{GL}(V_1^+) \times \mathrm{GL}(V_1^-) \times \mathrm{GL}(V_0^+) \times \mathrm{GL}(V_0^-) \to \mathrm{GL}(\mathscr{V})$$

be defined by

$$R(g_1^+, g_1^-, g_0^+, g_0^-) = \sigma(g_1^+)\widetilde{\sigma}(g_1^-)\omega(g_0^+)\widetilde{\omega}(g_0^-)\ , \tag{2.22}$$

where each of the factors σ, $\widetilde{\sigma}$, ω, and $\widetilde{\omega}$ operates on the corresponding factor in the tensor product $\mathscr{V}$ (and is trivial on all other factors). The character formulas (2.7), (2.9), (2.18), and (2.19) then combine to the following statement for the character $\mathrm{STr}_{\mathscr{V}} R = \mathrm{Tr}_{\mathscr{V}_0} R - \mathrm{Tr}_{\mathscr{V}_1} R$ of the $\mathbb{Z}_2$-graded representation space $\mathscr{V} = \mathscr{V}_0 \oplus \mathscr{V}_1$.

Proposition 2.3. — *The character of the* $\mathbb{Z}_2$*-graded representation R of Eq. (2.22) exists for* $(x,y,w,z) \in \mathrm{GL}(V_1^+) \times \mathrm{GL}(V_1^-) \times \mathrm{H}^<(V_0^+) \times \mathrm{H}^>(V_0^-)$ *where it has the value*

$$\mathrm{STr}_{\mathscr{V}} R(x,y,w,z) = \frac{\mathrm{Det}(\mathrm{Id}_{V_1^+} - x)\,\mathrm{Det}(y - \mathrm{Id}_{V_1^-})}{\mathrm{Det}(\mathrm{Id}_{V_0^+} - w)\,\mathrm{Det}(z - \mathrm{Id}_{V_0^-})}\ .$$

We now take the components of the $\mathbb{Z}_2$-graded vector space V of (2.20) to be

$$V_\tau^\pm = U_\tau^\pm \otimes \mathbb{C}^N \quad (\tau = 0, 1)$$

with Hermitian vector spaces $U_0^+ = U_1^+ = \mathbb{C}^p$ and $U_0^- = U_1^- = \mathbb{C}^q$. We also introduce $t := (t_1^+, t_1^-, t_0^+, t_0^-)$ where the components $t_\tau^\pm$ are diagonal operators in $\mathrm{End}(U_\tau^\pm)$,

$$\begin{aligned} t_1^+ &= \mathrm{diag}(\mathrm{e}^{\mathrm{i}\psi_1}, \ldots, \mathrm{e}^{\mathrm{i}\psi_p})\ , \qquad & t_1^- &= \mathrm{diag}(\mathrm{e}^{\mathrm{i}\psi_{p+1}}, \ldots, \mathrm{e}^{\mathrm{i}\psi_{p+q}})\ , \\ t_0^+ &= \mathrm{diag}(\mathrm{e}^{\phi_1}, \ldots, \mathrm{e}^{\phi_p})\ , & t_0^- &= \mathrm{diag}(\mathrm{e}^{\phi_{p+1}}, \ldots, \mathrm{e}^{\phi_{p+q}})\ , \end{aligned}$$

with complex parameters $\psi_1, \ldots, \phi_{p+q}$. Let $u \in \mathrm{U}_N$, the unitary group of the Hermitian vector space $\mathbb{C}^N$. With every pair (t,u) we associate an element $t \otimes u \in \mathbf{G}$ by

$$t \otimes u \equiv (t_1^+ \otimes u\,,\, t_1^- \otimes u\,,\, t_0^+ \otimes u\,,\, t_0^- \otimes u)\ .$$

Corollary 2.4. — *If the parameters are restricted to the range* $\mathfrak{Re}\,\phi_j < 0 < \mathfrak{Re}\,\phi_l$ *for* $j = 1, \ldots, p$ *and* $l = p+1, \ldots, p+q$*, the operator* $R(t \otimes u) : \mathscr{V} \to \mathscr{V}$ *is trace class and*

$$\mathrm{STr}_{\mathscr{V}} R(t \otimes u) = \prod_{j=1}^{p} \frac{\mathrm{Det}(\mathrm{Id}_N - \mathrm{e}^{\mathrm{i}\psi_j} u)}{\mathrm{Det}(\mathrm{Id}_N - \mathrm{e}^{\phi_j} u)} \prod_{l=p+1}^{p+q} \frac{\mathrm{Det}(\mathrm{e}^{\mathrm{i}\psi_l}\,\mathrm{Id}_N - \bar{u})}{\mathrm{Det}(\mathrm{e}^{\phi_l}\,\mathrm{Id}_N - \bar{u})}\ .$$

Proof. — By the restriction on the range of the parameters ϕ_j and ϕ_l, the operator $t_0^+ \otimes u$ lies in $\mathrm{H}^<(U_0^+ \otimes \mathbb{C}^N)$ and $t_0^- \otimes u$ lies in $\mathrm{H}^>(U_0^- \otimes \mathbb{C}^N)$. Therefore the formula of Prop. 2.3 applies with $(x,y,w,z) = (t_1^+ \otimes u\,,\, t_1^- \otimes u\,,\, t_0^+ \otimes u\,,\, t_0^- \otimes u)$. In particular, it follows that the supertrace of $R(t \otimes u)$ converges absolutely. By the elementary manipulation

$$\frac{\mathrm{Det}(\mathrm{e}^{\mathrm{i}\psi_l}\,u - \mathrm{Id}_N)}{\mathrm{Det}(\mathrm{e}^{\phi_l}\,u - \mathrm{Id}_N)} = \frac{\mathrm{Det}(\mathrm{e}^{\mathrm{i}\psi_l}\,\mathrm{Id}_N - \bar{u})}{\mathrm{Det}(\mathrm{e}^{\phi_l}\,\mathrm{Id}_N - \bar{u})}\ ,$$

the expression for $\mathrm{STr}_{\mathscr{V}} R(t \otimes u)$ is brought into the stated form. □

Remark. — Note that the expression for $\mathrm{STr}_{\mathscr{V}} R(t \otimes u)$ can be written as

$$\mathrm{STr}_{\mathscr{V}} R(t \otimes u) = \prod_{k=1}^{p+q} \frac{\mathrm{Det}(\mathrm{Id}_N - \mathrm{e}^{\mathrm{i}\psi_k} u)}{\mathrm{Det}(\mathrm{Id}_N - \mathrm{e}^{\phi_k} u)}\ .$$

This simpler expression hides the fact that complex conjugates $\bar{u}$ are lurking here by the implicit condition on the range of the parameters ϕ_k. Nonetheless, it is this last expression for $\mathrm{STr}_{\mathscr{V}} R(t \otimes u)$ which will be quoted in Prop. 2.10 below.

Cor. 2.4 says that (1.1) is a character. While of no immediate use by itself, this observation becomes a powerful fact when combined with the following message.

Since $(t,u) = (t, \mathrm{Id}) \cdot (\mathrm{Id}\,, u) = (\mathrm{Id}\,, u) \cdot (t, \mathrm{Id})$ and the sequence of maps

$$(t,u) \mapsto t \otimes u \mapsto R(t \otimes u)$$

is a homomorphism, we have a decomposition

$$R(t \otimes u) = \rho(t) r(u) = r(u) \rho(t)\ , \tag{2.23}$$

where the factors are

$$\rho(t) := R(t \otimes \mathrm{Id})\ , \quad r(u) := R(\mathrm{Id} \otimes u)\ . \tag{2.24}$$

It will turn out that these factors correspond to a so-called *Howe pair* $(\mathfrak{gl}_{n|n}\,, \mathrm{U}_N)$ where $n = p+q$. To appreciate this message and reap full benefit from its representation-theoretic impact, we are now going to broaden our framework.

2.5. Supersymmetric framework. — Recall from Sect. 1.2 that a $\mathbb{Z}_2$-graded complex vector space $V = V_1 \oplus V_0$ determines a complex Lie superalgebra $\mathfrak{gl}(V)$, whose Lie superbracket is given by the supercommutator $[X,Y] = XY - (-1)^{|X||Y|}YX$ for homogeneous elements $X, Y \in \mathfrak{gl}(V)$.

Another algebraic structure associated with $V = V_1 \oplus V_0$, or rather with

$$W = V \oplus V^* = (V_1 \oplus V_1^*) \oplus (V_0 \oplus V_0^*) =: W_1 \oplus W_0 \ ,$$

is the *Clifford-Weyl algebra* of W, which unifies the Clifford algebra $\mathfrak{c}(W_1)$ of the odd component W_1 with the Weyl algebra $\mathfrak{w}(W_0)$ of the even component W_0.

Definition 2.5. — *Let $Q : W \times W \to \mathbb{C}$ be the non-degenerate complex bilinear form for which $W = W_1 \oplus W_0$ is an orthogonal decomposition and which restricts to the canonical symmetric form S on $W_1 = V_1 \oplus V_1^*$ and the canonical alternating form A on $W_0 = V_0 \oplus V_0^*$. Then the Clifford-Weyl algebra of the $\mathbb{Z}_2$-graded vector space W is defined to be the associative algebra generated by $W \oplus \mathbb{C}$ with relations*

$$ww' - (-1)^{|w|\,|w'|} w'w = Q(w,w') \tag{2.25}$$

for homogeneous $w, w' \in W$. We denote it by $\mathfrak{q}(W)$ (with $\mathfrak{q}$ as in "quantum").

Remark. — Although $\mathfrak{q}(W)$ is primarily defined as an associative algebra, it carries a natural Lie superalgebra structure. Indeed, $\mathfrak{q}(W)$ as a vector space inherits from the Clifford algebra $\mathfrak{c}(W_1)$ a $\mathbb{Z}_2$-grading by the isomorphism

$$\begin{aligned} \mathfrak{q}(W) &\simeq \mathfrak{c}(W_1) \otimes \mathfrak{w}(W_0) \\ &\simeq \big(\mathfrak{c}^{\text{even}}(W_1) \otimes \mathfrak{w}(W_0)\big) \oplus \big(\mathfrak{c}^{\text{odd}}(W_1) \otimes \mathfrak{w}(W_0)\big) \ , \end{aligned}$$

and the supercommutator (1.18) determined by this $\mathbb{Z}_2$-grading is compatible with the defining relations (2.25).

There is a canonical way in which the Lie superalgebra $\mathfrak{gl}(V)$ is realized inside the Clifford-Weyl algebra $\mathfrak{q}(V \oplus V^*)$. To describe this realization, fix any homogeneous basis $\{e_i\}_{i=1,\ldots,\dim V}$ of V, and denote the dual basis of V^* by $\{f^i\}$. Then consider the isomorphism of $\mathbb{Z}_2$-graded vector spaces

$$\mathfrak{gl}(V) \to V \otimes V^* \ , \quad X \mapsto \sum (Xe_i) \otimes f^i \ . \tag{2.26}$$

Of course the tensor product $V \otimes V^*$ is equipped with the induced $\mathbb{Z}_2$-grading

$$\begin{aligned} (V \otimes V^*)_1 &= (V_0 \otimes V_1^*) \oplus (V_1 \otimes V_0^*) \ , \\ (V \otimes V^*)_0 &= (V_0 \otimes V_0^*) \oplus (V_1 \otimes V_1^*) \ , \end{aligned}$$

and with the bracket (for homogeneous elements $v, v' \in V$ and $\varphi, \varphi' \in V^*$)

$$[v \otimes \varphi, v' \otimes \varphi'] := \varphi(v')\, v \otimes \varphi' - (-1)^{|v\otimes\varphi|\,|v'\otimes\varphi'|} \varphi'(v)\, v' \otimes \varphi \ . \tag{2.27}$$

With these conventions one immediately verifies that

$$\Big[\sum (Xe_i) \otimes f^i , \sum (Ye_j) \otimes f^j\Big] = \sum ([X,Y]e_i) \otimes f^i \ .$$

Thus (2.26) is an isomorphism of Lie superalgebras.

Lemma 2.6. — *If $W = V \oplus V^*$, the linear mapping*

$$\mathfrak{gl}(V) \to \mathfrak{q}(W) , \quad X \mapsto \sum (X e_i) f^i \equiv \hat{X} ,$$

is a homomorphism of Lie superalgebras.

Proof. — We regard our mapping $\mathfrak{gl}(V) \to \mathfrak{q}(V \oplus V^*)$ as a sequence of two mappings: first is the map $\mathfrak{gl}(V) \to V \otimes V^*$ of (2.26), and this is followed by

$$V \otimes V^* \to \mathfrak{q}(V \oplus V^*) , \quad v \otimes \varphi \mapsto v\varphi .$$

Since the first map is an isomorphism, $X \mapsto \hat{X}$ is a homomorphism if $v \otimes \varphi \mapsto v\varphi$ is. To show that the latter is true, we use the defining Clifford-Weyl relations (2.25) to do the following computation (for $\mathbb{Z}_2$-homogeneous factors):

$$\begin{aligned} v\varphi v'\varphi' &= \varphi(v')\, v\varphi' + (-1)^{|\varphi|\,|v'|} v v' \varphi \varphi' \\ &= \varphi(v')\, v\varphi' + (-1)^{|\varphi|\,|v'| + |v|\,|v'| + |\varphi|\,|\varphi'|} v' v \varphi' \varphi . \end{aligned}$$

Subtraction of $(-1)^{(|v|+|\varphi|)(|v'|+|\varphi'|)} v'\varphi' v\varphi$ on both sides gives

$$[v\varphi , v'\varphi'] = \varphi(v')\, v\varphi' - (-1)^{(|v|+|\varphi|)(|v'|+|\varphi'|)} \varphi'(v)\, v'\varphi ,$$

which in fact agrees with the bracket (2.27) since $|v \otimes \varphi| = |v| + |\varphi|$ (mod 2). ☐

As an important corollary to Lem. 2.6 one concludes that every representation of the Clifford-Weyl algebra $\mathfrak{q}(W)$ for $W = V \oplus V^*$ induces a representation of the Lie superalgebra $\mathfrak{gl}(V)$. For this, one just sends $X \in \mathfrak{gl}(V)$ to its image in $\mathfrak{q}(W)$ under the mapping of Lem. 2.6, and then applies the given representation of $\mathfrak{q}(W)$.

We now adopt the following supersymmetric framework, which unifies and extends the algebraic structures underlying Sects. 2.1–2.3.

Definition 2.7. — *Given a $\mathbb{Z}_2$-graded vector space*

$$V = V_1 \oplus V_0 = (V_1^+ \oplus V_1^-) \oplus (V_0^+ \oplus V_0^-)$$

and its Clifford-Weyl algebra $\mathfrak{q}(W)$ for $W = V \oplus V^*$, define $\mathscr{A}_V \subset \mathfrak{q}(W)$ to be the graded-commutative subalgebra generated by the Q-isotropic vector subspace

$$(V_1^+ \oplus (V_1^-)^*) \oplus (V_0^+ \oplus (V_0^-)^*) \subset W .$$

On $\mathscr{A}_V$ let the operators of alternating multiplication ε, alternating contraction ι, symmetric multiplication μ, and symmetric contraction δ, be defined as usual. By the spinor-oscillator representation $\mathbf{q} : \mathfrak{q}(W) \to \mathrm{End}(\mathscr{A}_V)$ we mean the representation which is given by letting the elements of $W_\tau = V_\tau^+ \oplus (V_\tau^+)^* \oplus V_\tau^- \oplus (V_\tau^-)^*$ operate as

$$\begin{aligned} \mathbf{c}(v_1^+ + \varphi_1^+ + v_1^- + \varphi_1^-) &= \varepsilon(v_1^+) + \iota(\varphi_1^+) + \iota(v_1^-) + \varepsilon(\varphi_1^-) \quad (\tau = 1) , \\ \mathbf{w}(v_0^+ + \varphi_0^+ + v_0^- + \varphi_0^-) &= \mu(v_0^+) + \delta(\varphi_0^+) + \delta(v_0^-) - \mu(\varphi_0^-) \quad (\tau = 0) . \end{aligned}$$

Remark. — The operators $\varepsilon(v_1^+)$, $\iota(\varphi_1^+)$, and $\iota(v_1^-)$, $\varepsilon(\varphi_1^-)$ obey CAR (2.3) transcribed to $\mathscr{A}_V$; the operators $\mu(v_0^+)$, $\delta(\varphi_0^+)$, and $\delta(v_0^-)$, $-\mu(\varphi_0^-)$ obey CCR (2.12)

transcribed to $\mathscr{A}_V$; and the two sets of operators commute. Note that the algebra $\mathscr{A}_V$ is isomorphic to the spinor-oscillator module $\mathscr{V}$ of (2.21):

$$\mathscr{A}_V \simeq \mathscr{V} \simeq \sum \wedge^k (V_1^+ \oplus V_1^{-*}) \otimes \sum \mathrm{S}^l (V_0^+ \oplus V_0^{-*}) ,$$

and carries a natural $\mathbb{Z}$-grading by the total degree $n = k + l$. By this isomorphism, we will regard the linear operator $R(t \otimes u)$ of Cor. 2.4 from now on as an operator on $\mathscr{A}_V$.

Let $R_* : \mathfrak{gl}(V) \to \mathfrak{gl}(\mathscr{A}_V)$ denote the Lie superalgebra representation induced by $\mathbf{q} : \mathfrak{q}(W) \to \mathfrak{gl}(\mathscr{A}_V)$. It follows immediately from Def. 2.7 that

$$R_*|_{V_1^+ \to V_1^+} = \sigma , \quad R_*|_{V_1^- \to V_1^-} = \widetilde{\sigma} , \quad R_*|_{V_0^+ \to V_0^+} = \omega , \quad R_*|_{V_1^- \to V_1^-} = \widetilde{\omega} .$$

2.6. Howe duality and a consequence. — We now adapt the tensor-product situation of Cor. 2.4 to our expanded framework. Introducing a $\mathbb{Z}_2$-graded vector space U,

$$U = U_1 \oplus U_0 = (U_1^+ \oplus U_1^-) \oplus (U_0^+ \oplus U_0^-) , \tag{2.28}$$

with $U_0^+ = U_1^+ = \mathbb{C}^p$ and $U_0^- = U_1^- = \mathbb{C}^q$, we make in Def. 2.7 the identifications

$$V = U \otimes \mathbb{C}^N , \quad V_\tau^\pm = U_\tau^\pm \otimes \mathbb{C}^N \quad (\tau = 0, 1) . \tag{2.29}$$

This tensor-product decomposition of V gives rise to two distinguished subalgebras of $\mathfrak{gl}(V)$: there is a Lie superalgebra $\mathfrak{gl}(U)$ which acts on the $\mathbb{Z}_2$-graded vector space U as its fundamental module and operates trivially on the second factor, $\mathbb{C}^N$; and there is a Lie algebra $\mathfrak{gl}_N$ which acts on $\mathbb{C}^N$ and is trivial on the first factor, U.

R. Howe, in 1976, wrote an insightful article ”Remarks on Classical Invariant Theory” [**12**] which has much bearing on our present situation. The main concept is this.

Definition 2.8. — *Given an orthosymplectic $\mathbb{Z}_2$-graded vector space W, let (Γ, Γ') be a pair of Lie (sub-)superalgebras in $\mathfrak{osp}(W)$. Such a pair is called a reductive supersymmetric Howe dual pair if Γ' acts reductively on W and Γ is the centralizer of Γ' in $\mathfrak{osp}(W)$. The pair is called classical if Γ' is a classical Lie algebra.*

Proposition 2.9. — *If $V = U \otimes \mathbb{C}^N$ as above, the pair $(\mathfrak{gl}(U), \mathfrak{gl}_N)$ is a classical reductive supersymmetric Howe dual pair in $\mathfrak{osp}(V \oplus V^*)$.*

Proof. — The action of the classical Lie algebra $\mathfrak{gl}_N$ on $W := V \oplus V^*$ is reductive by the assumed tensor product structure $V = U \otimes \mathbb{C}^N$.

$\mathfrak{gl}_N$ and $\mathfrak{gl}(U)$ act on different factors of a tensor product $V = U \otimes \mathbb{C}^N$; therefore the two actions commute. Conversely, if $X \in \mathfrak{osp}(W)$ commutes with every $Y \in \mathfrak{gl}_N$, then X must be in $\mathfrak{gl}(V)$, since the two $\mathfrak{gl}_N$-representation spaces V and V^* belong to different isomorphism classes. Because $\mathbb{C}^N$ is $\mathfrak{gl}_N$-irreducible, it follows that $X \in \mathfrak{gl}(U)$. Thus $\mathfrak{gl}(U)$ indeed is the centralizer of $\mathfrak{gl}_N$ in $\mathfrak{osp}(W)$. ☐

Now set $n = p + q$ and note the identifications $U = \mathbb{C}^{n|n}$ and $\mathfrak{gl}(U) \simeq \mathfrak{gl}_{n|n}$. As before, let $R_* : \mathfrak{gl}(V) \to \mathfrak{gl}(\mathscr{A}_V)$ be the representation which is induced by the spinor-oscillator representation of the Clifford-Weyl algebra $\mathfrak{q}(V \oplus V^*)$ on $\mathscr{A}_V$. Given R_* and

the identification $V \simeq \mathbb{C}^{n|n} \otimes \mathbb{C}^N$, one immediately has a representation ρ_* of the Lie superalgebra $\mathfrak{gl}_{n|n} \hookrightarrow \mathfrak{gl}(V)$ on $\mathscr{A}_V$:

$$\rho_* : \mathfrak{gl}_{n|n} \to \mathfrak{gl}(\mathscr{A}_V) , \quad x \mapsto R_*(x \otimes \mathrm{Id}_N) . \tag{2.30}$$

Similarly, the $\mathfrak{gl}(V)$-representation R_* induces a representation r_* of $\mathfrak{gl}_N \hookrightarrow \mathfrak{gl}(V)$ by

$$r_* : \mathfrak{gl}_N \to \mathfrak{gl}(\mathscr{A}_V) , \quad y \mapsto R_*(\mathrm{Id}_{n|n} \otimes y) . \tag{2.31}$$

The latter arises also from the U_N-representation r of (2.24), by linearization at $u = \mathrm{Id}_N$ and complexification. This representation is unitary if $\mathscr{A}_V$ is given its canonical Hermitian structure, in which

$$\mu(v_0^+)^\dagger = \delta(c v_0^+) , \quad \varepsilon(v_1^+)^\dagger = \iota(c v_1^+) ,$$
$$\delta(v_0^-)^\dagger = \mu(c v_0^-) , \quad \iota(v_1^-)^\dagger = \varepsilon(c v_1^-) .$$

(See Sect. 2.2 for explanation in the bosonic case; the fermionic case is similar). Indeed, using these relations it is readily seen that $r_*(y)^\dagger = -r_*(y)$ for $y \in \mathfrak{u}_N = \mathrm{Lie}\,\mathrm{U}_N$. Note also that $y \in \mathfrak{u}_N$ acts on the spinor-oscillator module $\mathscr{A}_V$ by operators of type $\varepsilon\iota$, $\iota\varepsilon$, $\mu\delta$, and $\delta\mu$, none of which changes the degree of the $\mathbb{Z}$-graded algebra $\mathscr{A}_V$.

What we have at hand, then, is a Howe pair $(\mathfrak{gl}_{n|n}, \mathfrak{gl}_N)$ where the second member $\mathfrak{gl}_N = \mathfrak{u}_N + \mathrm{i}\mathfrak{u}_N$ acts on the spinor-oscillator module $\mathscr{A}_V$ by complex linear extension of the representation of its compact real form $\mathfrak{u}_N$. In such a situation one prefers the notation $(\mathfrak{gl}_{n|n}, \mathrm{U}_N) = (\Gamma, K)$ and speaks of a Howe dual pair with compact group K.

For present purposes, a major result of Howe's theory of dual pairs (Γ, K) with compact group K is Thm. 8 of [**12**], which is concerned with the question of how the spinor-oscillator representation decomposes with respect to the action of (Γ, K). In our case of $(\mathfrak{gl}_{n|n}, \mathrm{U}_N)$ represented by

$$\rho_* : \mathfrak{gl}_{n|n} \to \mathfrak{gl}(\mathscr{A}_V) , \quad r : \mathrm{U}_N \to \mathrm{U}(\mathscr{A}_V) ,$$

the theorem implies that the U_N-trivial isotypic component of $\mathscr{A}_V$, i.e., the space of U_N-invariant vectors in $\mathscr{A}_V$, is an *irreducible* representation space for $\mathfrak{gl}_{n|n}$.

To make use of this fact, recall from (2.24) the definition $\rho(t) = R(t \otimes \mathrm{Id}_N)$ for the diagonal operators $t = (t_1^+, t_1^-, t_0^+, t_0^-)$ specified for Cor. 2.4. If t is reinterpreted via

$$\mathrm{GL}(U_1^+) \times \mathrm{GL}(U_1^-) \times \mathrm{GL}(U_0^+) \times \mathrm{GL}(U_0^-) \hookrightarrow \mathrm{End}(U)$$

as a linear operator on U with logarithm $\ln t \in \mathfrak{gl}(U)$, the infinitesimal representation $\rho_* : \mathfrak{gl}_{n|n} \to \mathfrak{gl}(\mathscr{A}_V)$ is related to ρ by $\rho(t) = \mathrm{e}^{\rho_*(\ln t)}$.

Proposition 2.10. — *Denoting by du the Haar measure of $K \equiv \mathrm{U}_N$ with total mass equal to 1, define a function $t \to \chi(t)$ by*

$$t \mapsto \chi(t) := \int_{\mathrm{U}_N} \prod_{k=1}^{n} \frac{\mathrm{Det}(\mathrm{Id}_N - \mathrm{e}^{\mathrm{i}\psi_k} u)}{\mathrm{Det}(\mathrm{Id}_N - \mathrm{e}^{\phi_k} u)} \, du .$$

In the range of parameters $t = t(\psi_k, \phi_k)$ specified in Cor. 2.4, this function has an alternative expression as a character:

$$\chi(t) = \mathrm{STr}_{\mathscr{A}_V^K} \rho(t) \equiv \mathrm{STr}_{\mathscr{A}_V^K} R(t \otimes \mathrm{Id}_N) ,$$

which is an absolutely convergent sum over $\mathscr{A}_V^K$, the irreducible $\mathfrak{gl}_{n|n}$-representation space spanned by the K-invariant vectors in the spinor-oscillator module $\mathscr{A}_V$.

Proof. — Take the $\mathscr{A}_V$-supertrace of the factorization (2.23) and integrate over K:

$$\int_K \mathrm{STr}_{\mathscr{A}_V} R(t \otimes u)\, du = \int_K \mathrm{STr}_{\mathscr{A}_V} \rho(t) r(u)\, du\,.$$

On inserting the explicit form of $\mathrm{STr}_{\mathscr{A}_V} R(t \otimes u)$ stated in the remark after Cor. 2.4, the left-hand side of this equality turns into the defining expression for $\chi(t)$.

In the prescribed range for t, where $\mathrm{STr}_{\mathscr{A}_V} \rho(t) r(u)$ converges absolutely and uniformly in u, one may interchange the order of taking the trace over $\mathscr{A}_V$ and integrating over the compact group K. The result of doing the latter integral is another absolutely convergent sum, which coincides with $\mathrm{STr}_{\mathscr{A}_V^K} \rho(t)$. Indeed, since r is a unitary representation of K on $\mathscr{A}_V$, Haar-averaging the operator $r(u)$ over $u \in K$ yields the projector onto $\mathscr{A}_V^K$, the subspace of K-invariants in $\mathscr{A}_V$.

The irreducibility of $\mathscr{A}_V^K$ w.r.t. the $\mathfrak{gl}_{n|n}$-action is implied by Thm. 8 of [**12**]. □

Remark. — Since the K-action on $\mathscr{A}_V$ preserves the $\mathbb{Z}$-grading of this graded-commutative algebra, the subalgebra of K-invariants $\mathscr{A}_V^K$ is still $\mathbb{Z}$-graded. For the same reason, the $\mathbb{Z}_2$-grading of $\mathscr{A}_V$ defines a $\mathbb{Z}_2$-grading of $\mathscr{A}_V^K$.

Our function $t \mapsto \chi(t)$ is identical to the function on the left-hand side of the statement of Thm. 1.1. We have thus established that left-hand side to be a character associated with the irreducible representation $(\mathscr{A}_V^K, \rho_*)$ of $\mathfrak{gl}_{n|n}$. This completes the first step of the programme outlined in Sect. 1.4.

We shall see that the character $\chi(t)$ is expressed by a generalization of the Weyl character formula for irreducible representations of compact Lie groups. If that generalization were available in the published literature, we would already be done and this paper could end right here. However, the required generalization apparently has not been worked out before; it will keep us busy for another three main sections.

2.7. Weight expansion of χ. — We begin our investigation of the character $\chi(t)$ by gathering some standard facts about the Lie superalgebra $\mathfrak{g} \equiv \mathfrak{gl}(U) = \mathfrak{gl}_{n|n}$.

Let $\mathfrak{h} \subset \mathfrak{g}$ denote the maximal Abelian subalgebra which is generated over $\mathbb{C}$ by the projection operators $E_{\tau,i}^{\pm} \in \mathrm{End}(U_\tau^{\pm})$ of Cor. 2.4. Although the variables $\psi_1, \ldots, \phi_n$ were originally introduced as *parameters*, it is now natural to reinterpret them as *linear coordinate functions* on $\mathfrak{h}$. Adopting this new perspective we expand $H \in \mathfrak{h}$ as

$$H = \mathrm{diag}\big(\mathrm{i}\psi_1(H), \ldots, \mathrm{i}\psi_{p+q}(H), \phi_1(H), \ldots, \phi_{p+q}(H)\big)\,.$$

In Lie theory, the non-zero eigenvalues, α, of the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ are called the *roots* of the pair $(\mathfrak{h}, \mathfrak{g})$. In the present context, a root is called *even* or *odd* depending on whether the eigenvector X in the eigenvalue equation $[H, X] = \alpha(H) X$ is an even or odd element of $\mathfrak{g}$. The even roots of our Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(U)$ are $\mathrm{i}\psi_k - \mathrm{i}\psi_{k'}$ and $\phi_k - \phi_{k'}$ (for $k \neq k'$), while the odd roots are $\mathrm{i}\psi_k - \phi_{k'}$ and $\phi_k - \mathrm{i}\psi_{k'}$ (for any k, k').

By general principles, if α is a root, then so is $-\alpha$. Let us fix a system of *positive roots*, Δ^+. For that, we arrange the coordinate functions as an ordered set:

$$\phi_1, \ldots, \phi_p, \mathrm{i}\psi_1, \ldots, \mathrm{i}\psi_p, \mathrm{i}\psi_{p+1}, \ldots, \mathrm{i}\psi_{p+q}, \phi_{p+1}, \ldots, \phi_{p+q}, \tag{2.32}$$

and take Δ^+ to be the set of differences $x-y$ where x and y are any two entries from this sequence subject to the requirement that x occurs later than y. To illustrate: $\phi_2 - \phi_1$ and $\mathrm{i}\psi_1 - \phi_1$ are two examples of positive roots. If $\mathfrak{g}^\alpha \subset \mathfrak{g}$ denotes the root space of the root α, i.e., the $\mathrm{ad}(\mathfrak{h})$-eigenspace with eigenvalue α, then

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ , \qquad \mathfrak{n}^+ = \sum_{\alpha\in\Delta^+} \mathfrak{g}^\alpha , \quad \mathfrak{n}^- = \sum_{\alpha\in\Delta^+} \mathfrak{g}^{-\alpha} .$$

While all this is standard and applies to semisimple Lie algebras and superalgebras in general, the situation at hand is actually ruled by a coarser structure. Rearranging the components of the vector space U of (2.28) as

$$U = U^+ \oplus U^- , \quad U^\pm = U_1^\pm \oplus U_0^\pm , \tag{2.33}$$

our Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(U)$ is $\mathbb{Z}$-graded by a direct-sum decomposition

$$\mathfrak{g} = \mathfrak{g}^{(-2)} \oplus \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(2)} , \tag{2.34}$$

where the middle summand is a Lie (sub-)superalgebra in $\mathfrak{g}$:

$$\mathfrak{g}^{(0)} = \mathfrak{gl}(U^+) \oplus \mathfrak{gl}(U^-) = \mathfrak{gl}_{p|p} \oplus \mathfrak{gl}_{q|q} , \tag{2.35}$$

while the first and last summands are $\mathbb{Z}_2$-graded vector spaces

$$\mathfrak{g}^{(-2)} = \mathrm{Hom}(U^+, U^-) , \quad \mathfrak{g}^{(2)} = \mathrm{Hom}(U^-, U^+) , \tag{2.36}$$

which are $\mathfrak{g}^{(0)}$-modules by the adjoint action. Note the inclusions

$$\mathfrak{h} \subset \mathfrak{g}^{(0)} , \quad \mathfrak{g}^{(-2)} \subset \mathfrak{n}^+ , \quad \mathfrak{g}^{(2)} \subset \mathfrak{n}^- . \tag{2.37}$$

The first one is obvious. The last two follow from our choice of positive root system by noticing that the functions $\psi_1, \ldots, \phi_{p+q}$ have been arranged in (2.32) so that the first $2p$ of them vanish on $\mathfrak{h} \cap \mathfrak{gl}(U^-)$ and the last $2q$ vanish on $\mathfrak{h} \cap \mathfrak{gl}(U^+)$. (The apparent sign inconsistency is forced by our desire to follow mathematical conventions and treat the *lowest*-degree subspace of $\mathscr{A}_V^K$ as a *highest*-weight space.)

The decomposition (2.34) reflects the way in which $\mathfrak{gl}(U) \hookrightarrow \mathfrak{gl}(V) = \mathfrak{gl}(U \otimes \mathbb{C}^N)$ is represented on the spinor-oscillator module $\mathscr{A}_V$ of Def. 2.7: the elements in $\mathfrak{g}^{(2)}$ act as operators of type $\varepsilon\varepsilon$, $\varepsilon\mu$, or $\mu\mu$, which raise the degree in $\mathscr{A}_V$ and $\mathscr{A}_V^K$ by two; the elements of $\mathfrak{g}^{(-2)}$ act as operators of type $\iota\iota$, $\iota\delta$, or $\delta\delta$, which lower the degree by two; and $\mathfrak{g}^{(0)}$ is represented by degree-preserving operators of type $\varepsilon\iota$, $\varepsilon\delta$, $\mu\iota$, or $\mu\delta$.

The following proposition means that $(\mathscr{A}_V^K, \rho_*)$ is a highest-weight (actually, lowest-weight) representation of $\mathfrak{gl}_{n|n}$. As usual, the eigenvalues and eigenspaces of the $\mathfrak{h}$-action by ρ_* on $\mathscr{A}_V^K$ are called the *weights* and *weight spaces* of the representation.

Notice that the subspace of constants $\mathbb{C} \subset \mathscr{A}_V$ (the "vacuum" in physics language) is invariant under the action of $K = \mathrm{U}_N$ and hence lies in $\mathscr{A}_V^K$.

Proposition 2.11. — *The one-dimensional subspace of constants $\mathbb{C} \subset \mathscr{A}_V^K$ is a cyclic subspace for $\mathscr{A}_V^K$ under the action of the algebra $\mathfrak{g}^{(2)}$ of raising operators. It is stabilized by the subalgebra $\mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(-2)}$. The weight of this subspace is*

$$\lambda_N = N \sum_{l=p+1}^{p+q} (\mathrm{i}\psi_l - \phi_l) \, .$$

Proof. — The degree-zero subspace of $\mathscr{A}_V$ is one-dimensional, being given just by the constants, and this remains of course true on passing to $\mathscr{A}_V^K$. Because zero is the smallest possible degree and the action of $\mathfrak{g}^{(k)}$ changes the degree by k units, $\mathfrak{g}^{(-2)}$ annihilates the degree-zero subspace $\mathbb{C} \subset \mathscr{A}_V^K$, and $\mathfrak{g}^{(0)}$ stabilizes it.

The cyclic property, i.e., the fact that every vector of $\mathscr{A}_V^K$ is generated by the successive application of raising operators in $\mathfrak{g}^{(2)}$, is asserted by Thm. 9 (iii) of [**12**].

Let now $z \in \mathbb{C} \subset \mathscr{A}_V^K$ be any non-zero constant. Applying to it the operator $\rho_*(H)$,

$$\begin{aligned} \rho_*(H)\, z \;=\; \sum_{a=1}^{N} \sum_{l=1}^{q} \big(& \mathrm{i}\psi_{l+p}(H)\, \iota(e_{1,l}^- \otimes e_a)\, \varepsilon(f_-^{1,l} \otimes f^a) \\ & - \phi_{l+p}(H)\, \delta(e_{0,l}^- \otimes e_a)\, \mu(f_-^{0,l} \otimes f^a) \big)\, z \, , \end{aligned}$$

and using the relations $\iota(v)\varepsilon(\varphi)\cdot 1 = \varphi(v)$ and $\delta(v)\mu(\varphi)\cdot 1 = \varphi(v)$, one sees that $z \in \mathbb{C}$ is an eigenvector with the stated eigenvalue $\lambda_N(H)$ for any $H \in \mathfrak{h}$. $\square$

Now define a real form $\mathfrak{h}'_{\mathbb{R}}$ of $\mathfrak{h}$ by demanding that all of the $\mathbb{C}$-linear coordinate functions $\mathrm{i}\psi_k : \mathfrak{h} \to \mathbb{C}$ and $\phi_k : \mathfrak{h} \to \mathbb{C}$ take *imaginary* values on $\mathfrak{h}'_{\mathbb{R}}$. The diagonal transformations $\exp H$ for $H \in \mathfrak{h}'_{\mathbb{R}}$ then form a compact Abelian group, and the $\mathfrak{gl}_{n|n}$-module $\mathscr{A}_V^K$ decomposes as an orthogonal direct sum of weight spaces V_γ with respect to the unitary action $\exp H \mapsto \exp \rho_*(H)$:

$$\mathscr{A}_V^K = \bigoplus_{\text{weights } \gamma} V_\gamma \, . \tag{2.38}$$

Proposition 2.12. — *Expressing the weights γ of the decomposition (2.38) as*

$$\gamma = \sum_{k=1}^{p+q} (\mathrm{i} m_k \psi_k - n_k \phi_k) \, ,$$

their coefficients are integers in the range

$$n_j \le 0 \le m_k \le N \le n_l \quad (1 \le j \le p < l \le p+q) \, .$$

Proof. — From general theory [**18**] one knows that the weights of an irreducible representation with highest weight λ are of the form $\lambda + \sum c_\alpha \alpha$ where the coefficients c_α are non-negative integers, and the sum over roots excludes those α whose root vectors annihilate or stabilize the highest weight space. In the present case, where the highest weight space is annihilated and stabilized by $\mathfrak{g}^{(-2)}$ resp. $\mathfrak{g}^{(0)}$, we introduce the set

$$\Delta_\lambda^+ := \{ \alpha \in \Delta^+ \mid \mathfrak{g}^{-\alpha} \subset \mathfrak{g}^{(2)} \} \, , \tag{2.39}$$

and the weights of the representation $(\mathscr{A}_V^K, \rho_*)$ are then of the form

$$\gamma = \lambda_N - \sum_{\alpha \in \Delta_\lambda^+} c_\alpha \, \alpha$$

with integers $c_\alpha \geq 0$. Since $\mathfrak{g}^{(2)} = \mathrm{Hom}(U^-, U^+)$, the choice of ordering in the list (2.32) means that the set Δ_λ^+ consists of the roots obtained by subtracting an entry in the second half of the list from an entry in the first half. These are the roots

$$\phi_l - \phi_j \,, \quad \mathrm{i}\psi_l - \mathrm{i}\psi_j \,, \quad \phi_l - \mathrm{i}\psi_j \,, \quad \mathrm{i}\psi_l - \phi_j \quad (1 \leq j \leq p < l \leq p+q) \,. \qquad (2.40)$$

Subtracting from the highest weight λ_N a linear combination of these roots with positive integers as coefficients, one gets weights γ with coefficients in the range $n_j \leq 0$, $n_l \geq N$, $m_j \geq 0$, and $m_l \leq N$ (for j,l still subject to $1 \leq j \leq p < l \leq p+q$).

The stronger condition $0 \leq m_k \leq N$ (for all $k = 1, \ldots, p+q$) is best seen by going back to the integral formula for the character χ of $(\mathscr{A}_V^K, \rho_*)$ in Prop. 2.10. Indeed, since $\mathrm{Det}(\mathrm{Id}_N - \mathrm{e}^{\mathrm{i}\psi_k} u)$ is a polynomial in $\mathrm{e}^{\mathrm{i}\psi_k}$ of degree N, the weight-space decomposition of $\mathscr{A}_V^K$ must be such that only the powers $\mathrm{e}^{\mathrm{i}m_k\psi_k}$ with $0 \leq m_k \leq N$ appear. □

Remark. — The weights γ of the weight-space decomposition (2.38) are *analytically integral*, i.e., whenever $H \in \mathfrak{h}$ is such that $\exp H \in \mathrm{End}(U)$ is unity, then $\mathrm{e}^{\gamma(H)} = 1$.

Let now Γ_λ denote the set of weights of our representation $(\mathscr{A}_V^K, \rho_*)$ with highest weight λ_N and $\mathbb{Z}_2$-grading $\mathscr{A}_V^K = (\mathscr{A}_V^K)_0 \oplus (\mathscr{A}_V^K)_1$. For $\gamma \in \Gamma_\lambda$ put $|\gamma| := 0$ if $V_\gamma \subset (\mathscr{A}_V^K)_0$ and $|\gamma| := 1$ if $V_\gamma \in (\mathscr{A}_V^K)_1$. (One of these must be true, as roots are either even or odd and weights are generated additively and integrally from roots.)

Corollary 2.13. — *The $\mathfrak{gl}_{n|n}$-character χ has an expansion as a sum over weights,*

$$\chi(t) = \sum\nolimits_{\gamma \in \Gamma_\lambda} (-1)^{|\gamma|} \dim(V_\gamma)\, \mathrm{e}^{\gamma(\ln t)} \,,$$

which converges absolutely in the domain for t defined by

$$\mathfrak{Re}\, \phi_j(\ln t) < 0 < \mathfrak{Re}\, \phi_l(\ln t) \quad (1 \leq j \leq p < l \leq p+q) \,.$$

The coefficients m_k, n_k of the weights $\gamma \in \Gamma_\lambda$ lie in the range specified by Prop. 2.12.

Proof. — Based on Prop. 2.10, compute $\chi(t) = \mathrm{STr}_{\mathscr{A}_V^K}\, \rho(t)$ by using the weight-space decomposition of $\mathscr{A}_V^K$ and the fact that $\rho(t)v = \mathrm{e}^{\rho_*(\ln t)}v = \mathrm{e}^{\gamma(\ln t)}v$ for $v \in V_\gamma$. The sign factor arises because one is computing a supertrace and the vector space V_γ has parity $(-1)^{|\gamma|}$. Since $|\mathrm{e}^{\gamma(\ln t)}| = \mathrm{e}^{\mathfrak{Re}\,\gamma(\ln t)}$, absolute convergence in the stated domain for t follows from the limited range of the coefficients m_k, n_k as given in Prop. 2.12. □

Remark. — Finding the multiplicities $\dim(V_\gamma)$ is a combinatorial task which we shall side-step in this paper by digging deeper into the structure of the problem.

3. Background material from superanalysis

Our task is to compute the character $t \mapsto \chi(t)$ and show that it is given by the right-hand side in the statement of Thm. 1.1. For that purpose it is not prudent to cling to the limited view of χ as a function of the diagonal operators t. Rather, to get enough analytical control we are going to exploit the fact that χ extends to a function on (a certain subspace of) the Lie supergroup $\mathrm{GL}_{n|n}$. The desired statement will then follow rather easily from some standard results in superanalysis.

Since superanalysis is not a widely known subject, we inject the present section to collect most of the necessary material for the convenience of the reader. (Experts in superanalysis should of course skip over this section.)

3.1. The vector bundle underlying a classical Lie supergroup. — Lie supergroups have been defined and studied by Berezin [**2**]. Here we will review a simplified definition, which uses nothing but canonical constructions in geometry.

Recall from Sect. 1.2 the notion of a Lie superalgebra $\mathfrak{g}$. In the sequel we will be concerned only with $\mathfrak{g} = \mathfrak{gl}$ and $\mathfrak{g} = \mathfrak{osp}$. Our number field $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$.

Let now G be a Lie group with Lie algebra $\mathrm{Lie}(G) = \mathfrak{g}_0 \subset \mathfrak{g}$. In the case of $\mathfrak{g} = \mathfrak{gl}_{p|q}$ we take $G \simeq \mathrm{GL}(\mathbb{K}^p) \times \mathrm{GL}(\mathbb{K}^q)$, i.e., the set of linear transformations

$$g = \begin{pmatrix} g_1 & 0 \\ 0 & g_0 \end{pmatrix} ,$$

with $g_1 \in \mathrm{GL}(\mathbb{K}^p)$ and $g_0 \in \mathrm{GL}(\mathbb{K}^q)$. If $\mathfrak{g} = \mathfrak{osp}_{p|q}$ we take $G \simeq \mathrm{SO}(\mathbb{K}^p) \times \mathrm{Sp}(\mathbb{K}^q)$.

To construct the Lie supergroups $\mathfrak{G} = \mathrm{GL}_{p|q}$ and $\mathfrak{G} = \mathrm{OSp}_{p|q}$ one should in principle invoke a Grassmann algebra of anti-commuting parameters to exponentiate the odd part $\mathfrak{g}_1$ of the Lie superalgebra $\mathfrak{g}$ [**2**]. The difficult step in this approach is to explain exactly what is meant by the group multiplication law. We will therefore adopt the viewpoint offered by Deligne and Morgan in their Notes on Supersymmetry (following J. Bernstein) [**10**]; see also [**20**]. In that approach, supergroup multiplication is described only at the infinitesimal level, as follows.

With the aim of constructing a certain important vector bundle (see below), consider the mapping

$$P := G \times G \to G , \quad (h', h) \mapsto h' h^{-1} ,$$

and view P as a G-principal bundle by the right G-action $(h', h) \mapsto (h'g, hg)$. The structure group G of P also acts on the odd part $\mathfrak{g}_1$ of the Lie superalgebra $\mathfrak{g}$ by the adjoint representation, Ad; in the cases at hand the adjoint action of G on $\mathfrak{g}$ is

$$\mathrm{Ad} \begin{pmatrix} g_1 & 0 \\ 0 & g_0 \end{pmatrix} : \quad \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & \mathsf{D} \end{pmatrix} \mapsto \begin{pmatrix} g_1 \mathsf{A} g_1^{-1} & g_1 \mathsf{B} g_0^{-1} \\ g_0 \mathsf{C} g_1^{-1} & g_0 \mathsf{D} g_0^{-1} \end{pmatrix} ,$$

and the G-action on $\mathfrak{g}_1$ is obtained by restricting to $\mathsf{A} = \mathsf{D} = 0$.

It is then a standard construction of differential geometry to associate to the principal bundle $P \to G$ a vector bundle $F \to G$ with total space

$$F = P \times_G \mathfrak{g}_1 ,$$

whose elements are the orbits of the diagonal right G-action on $P \times \mathfrak{g}_1$. In other words, the fibre of F over a point $x = h' h^{-1} \in G$ is the vector space $F_x \simeq \mathfrak{g}_1$ of G-orbits

$$[(h', h); Y] \equiv \{ h'g , hg , \mathrm{Ad}(g^{-1}) Y \mid g \in G \} \quad (Y \in \mathfrak{g}_1) .$$

The adjoint representation Ad of G on $\mathfrak{g}_1$ induces a representation $(\mathrm{Ad}^*)^{-1}$ of G on $\wedge(\mathfrak{g}_1^*)$, the exterior algebra of the dual of $\mathfrak{g}_1$. Thus, along with the vector bundle $F \to G$ we have a vector bundle

$$\wedge(F^*) \to G ,$$

whose fibre over a point $h'h^{-1} = x \in G$ is $\wedge F_x^* \simeq \wedge \mathfrak{g}_1^*$, i.e., the vector space of G-orbits

$$[(h',h)\,;a] \equiv \{h'g\,,hg\,,\mathrm{Ad}^*(g)a \mid g \in G\} \quad (a \in \wedge \mathfrak{g}_1^*)\ .$$

Of course the exterior multiplication in the fibre $\wedge F_x^*$ over $x = h'h^{-1}$ is defined by

$$[(h',h)\,;a] \wedge [(h',h)\,;b] = [(h',h)\,;a \wedge b] \qquad (a,b \in \wedge \mathfrak{g}_1^*)\ .$$

Given the vector bundle $\wedge F^* \to G$, we will now consider the graded-commutative algebra, $\mathscr{F}$, of smooth sections of $\wedge F^*$:

$$\mathscr{F} := \Gamma(G, \wedge F^*)\ .$$

Here the basic idea is that a Lie group (just like any manifold) can be described either by its points, or by the algebra of its functions (more precisely: the sheaf of algebras). In the case of a Lie supergroup, the latter viewpoint is the good one, i.e., one uses a description by the algebra of superfunctions, and this algebra is none other than $\mathscr{F}$.

3.2. $\mathfrak{g}$-action on a Lie supergroup. — A very useful fact about $\mathscr{F}$ is that its elements are in one-to-one correspondence with maps $\Phi \in \mathrm{C}^\infty(P, \wedge \mathfrak{g}_1^*)^G$, i.e., with smooth functions Φ on $P = G \times G$ that take values in $\wedge(\mathfrak{g}_1^*)$ and are equivariant w.r.t. G :

$$\Phi(h',h) = \mathrm{Ad}^*(g^{-1})\,\Phi(h'g,hg) \quad (h',h,g \in G)\ .$$

Indeed, such a function Φ determines a unique section $s \in \mathscr{F}$ by

$$s\big(h'h^{-1}\big) = [(h',h)\,;\Phi(h',h)] = [(h'g,hg)\,;\Phi(h'g,hg)]\ .$$

In the sequel we assume the identification $\mathrm{C}^\infty(P, \wedge \mathfrak{g}_1^*)^G \simeq \mathscr{F}$ by this bijection to be understood, and we will often abuse language to call $\Phi \in \mathrm{C}^\infty(P, \wedge \mathfrak{g}_1^*)^G$ a section of $\mathscr{F}$.

The algebra $\mathscr{F}$ has a natural $\mathbb{Z}_2$-grading $\mathscr{F} = \mathscr{F}_0 \oplus \mathscr{F}_1$ by

$$\mathscr{F}_0 = \bigoplus\nolimits_{\ell \text{ even}} \Gamma(G, \wedge^\ell F^*)\ , \quad \mathscr{F}_1 = \bigoplus\nolimits_{\ell \text{ odd}} \Gamma(G, \wedge^\ell F^*)\ .$$

The numerical (or $\mathbb{C}$-number) part of a section $s \in \mathscr{F}$ will be denoted by $\mathrm{num}(s)$.

An *even derivation* of $\mathscr{F}$ is a first-order differential operator $D : \mathscr{F}_0 \to \mathscr{F}_0$ and $D : \mathscr{F}_1 \to \mathscr{F}_1$. An *odd derivation* is a first-order differential operator $D : \mathscr{F}_0 \to \mathscr{F}_1$ and $D : \mathscr{F}_1 \to \mathscr{F}_0$ satisfying the anti-Leibniz rule $D(st) = (Ds)\,t + (-1)^{|s|} s\,Dt$.

3.2.1. The action of $\mathfrak{g}_0$. — We are now ready to specify half of the Lie supergroup structure. The Lie group G acts on $\mathscr{F} \simeq \mathrm{C}^\infty(P, \wedge \mathfrak{g}_1^*)^G$ in the obvious manner: there is a G-action on the left by

$$(L_g\Phi)(h,h') = \Phi(g^{-1}h,h')\ ,$$

and a G-action on the right by

$$(R_g\Phi)(h,h') = \Phi(h,g^{-1}h')\ .$$

Specializing these to the infinitesimal level we get the corresponding canonical actions of the Lie algebra $\mathfrak{g}_0$ by even derivations of $\mathscr{F}$:

$$\widehat{X}^L := \frac{d}{dt} L_{\exp(tX)}\Big|_{t=0}\ , \quad \widehat{X}^R := \frac{d}{dt} R_{\exp(tX)}\Big|_{t=0}\ .$$

Note the representation property $[\widehat{X}_1^L, \widehat{X}_2^L] = \widehat{[X_1, X_2]}^L$ and $[\widehat{X}_1^R, \widehat{X}_2^R] = \widehat{[X_1, X_2]}^R$.

3.2.2. Grassmann envelope. — The odd part $\mathfrak{g}_1$ of the Lie superalgebra $\mathfrak{g}$ acts on the graded-commutative algebra $\mathscr{F}$ by odd derivations. To describe this action one needs the notion of Grassmann envelope of a Lie superalgebra.

First, consider $\mathrm{End}(V)$ for some basic $\mathbb{Z}_2$-graded vector space V. Picking any parameter space $\mathbb{K}^d$ and denoting its exterior algebra by

$$\Omega = \Omega_0 \oplus \Omega_1 = \wedge^{\mathrm{even}}(\mathbb{K}^d) \oplus \wedge^{\mathrm{odd}}(\mathbb{K}^d) \, ,$$

the *Grassmann envelope* of $\mathrm{End}(V)$ by Ω is the vector space

$$\mathrm{End}_\Omega(V) = \bigoplus\nolimits_{\tau=0,1} \Omega_\tau \otimes \mathrm{End}(V)_\tau \, .$$

Thus an element $\xi \in \mathrm{End}_\Omega(V)$ consists of an even part with commuting parameters as coefficients and an odd part with anti-commuting parameters as coefficients. When $\mathrm{End}(V)$ is realized by matrices, ξ is called a *supermatrix*.

There exists more than one associative multiplication law to turn the vector space $\mathrm{End}_\Omega(V)$ into an associative algebra. To achieve consistency, we have to adopt here the convention of a Grassmann envelope of the *second kind* [**2**]. This means that the associative product for homogeneous $\alpha, \beta \in \Omega$ and $X, Y \in \mathrm{End}(V)$ is defined by

$$(\alpha \otimes X)\,(\beta \otimes Y) := (-1)^{|X|\,|\beta|} \alpha\beta \otimes XY \, ,$$

where we follow the general sign rule [**10**] which tells us to insert a minus sign (due to reversing the order) when both X and β are odd.

Now transcribe this construction to the case of a Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(V)$ or $\mathfrak{g} = \mathfrak{osp}(V)$. The Grassmann envelope of $\mathfrak{g}$ by Ω then is the vector space

$$\tilde{\mathfrak{g}}(\Omega) := \bigoplus\nolimits_{\tau=0,1} \Omega_\tau \otimes \mathfrak{g}_\tau \, ,$$

and this carries the natural structure of a Lie algebra with commutator bracket defined (still for homogeneous $\alpha, \beta \in \Omega$ and $X, Y \in \mathfrak{g}$) by

$$\begin{aligned} [\alpha \otimes X, \beta \otimes Y] \;&:=\; (-1)^{|X||\beta|} \alpha\beta \otimes XY - (-1)^{|Y||\alpha|} \beta\alpha \otimes YX \\ &=\; (-1)^{|\alpha||\beta|} \alpha\beta \otimes (XY - (-1)^{|X||Y|} YX) = \beta\alpha \otimes [X,Y] \, , \end{aligned}$$

where $[X,Y]$ is the bracket in $\mathfrak{g}$. Note that our Lie group G naturally acts on the Grassmann envelope $\tilde{\mathfrak{g}}(\Omega)$ by the adjoint action $\mathrm{Ad}(g)(\alpha \otimes X) := \alpha \otimes \mathrm{Ad}(g)X$.

3.2.3. The action of $\mathfrak{g}_1$. — Now fix some basis $\{\mathsf{F}_i\}$ of $\mathfrak{g}_1$, and denote the dual basis of $\mathfrak{g}_1^*$ by $\{\varphi^i\}$. (The construction to be made does not depend on which basis is chosen.) Then let $\xi \in \mathfrak{g}_1^* \otimes \mathfrak{g}_1$ be the tautological object

$$\xi = \sum \varphi^i \otimes \mathsf{F}_i \, ,$$

and repeat the change of interpretation that was made for the operator B in Sect. 1.2: the $\mathsf{F}_i \in \mathfrak{g}_1$ are viewed as linear transformations and multiplied as linear transformations; whereas the φ^i, being linear functions on the odd space $\mathfrak{g}_1$, are multiplied via the wedge product and viewed as a set of generators of the exterior algebra $\wedge(\mathfrak{g}_1^*)$.

Notice that, when $\mathfrak{g} = \mathfrak{gl}(V)$ or $\mathfrak{g} = \mathfrak{osp}(V)$ acts on its fundamental module V, the exponential e^{ξ} makes sense as an element of the Grassmann envelope $\mathrm{End}_{\wedge \mathfrak{g}_1^*}(V)$:

$$\mathrm{e}^{\xi} = \mathrm{e}^{\sum \varphi^i \otimes \mathsf{F}_i} = 1 \otimes \mathrm{Id}_V + \sum \varphi^i \otimes \mathsf{F}_i + \tfrac{1}{2} \sum \varphi^j \varphi^i \otimes \mathsf{F}_i \mathsf{F}_j + \ldots .$$

Further, the mapping $P = G \times G \to \mathrm{End}_{\wedge \mathfrak{g}_1^*}(V)$ by $(h', h) \mapsto (1 \otimes h')\, \mathrm{e}^{\sum \varphi^i \otimes \mathsf{F}_i} (1 \otimes h)^{-1}$ is equivariant with respect to G. Indeed, dropping the trivial factors of unity, one has

$$h'\, \mathrm{e}^{\sum \varphi^i \otimes \mathsf{F}_i} h^{-1} = (h'g)\, \mathrm{e}^{\sum \varphi^i \otimes \mathrm{Ad}(g^{-1}) \mathsf{F}_i} (hg)^{-1} ,$$

and since $\xi = \sum \varphi^i \otimes \mathsf{F}_i$ is G-invariant by definition, one may trade the right G-action $g \mapsto \mathrm{Ad}(g^{-1})$ on $\mathfrak{g}_1$ for the induced action $g \mapsto \mathrm{Ad}^*(g^{-1})$ on g_1^*.

Using the notion of Grassmann envelope for $\mathfrak{g}$, we are now going to describe the action $Y \mapsto \widehat{Y}^R$ of an odd generator $Y \in \mathfrak{g}_1$ on the sections $\Phi \in \mathscr{F}$. For the purpose of differentiation one extra anti-commuting parameter is needed, and we therefore enlarge the parameter Grassmann algebra to be

$$\Omega = \wedge(\mathfrak{g}_1^* \oplus \mathbb{K}) = \oplus_k \Omega^k .$$

Let $\sigma \in \mathbb{K}$ denote a generator of the second summand, $\mathbb{K}$, and fix some basis $\{\mathsf{E}_j\}$ of the Lie algebra $\mathfrak{g}_0 \subset \mathfrak{g}$. The main step toward defining $\widehat{Y}^R$ is the following lemma.

***Lemma 3.1* (Berezin).** — *There exist even- and odd-type functions*

$$\begin{aligned} \alpha^i :\ & \mathfrak{g}_1 \times G \to \wedge^{\mathrm{even}}(\mathfrak{g}_1^*) , \qquad (Y, g) \mapsto \alpha_Y^i(g) \quad (i = 1, \ldots, \dim \mathfrak{g}_1) , \\ \beta^j :\ & \mathfrak{g}_1 \times G \to \wedge^{\mathrm{odd}}(\mathfrak{g}_1^*) , \qquad (Y, g) \mapsto \beta_Y^j(g) \quad (j = 1, \ldots, \dim \mathfrak{g}_0) \end{aligned}$$

with linear (resp. differentiable) dependence on their first (resp. second) argument, such that for all $g', g \in G$ one has

$$(g' \mathrm{e}^{\xi} g^{-1}) \mathrm{e}^{\sigma \otimes Y} \equiv g' \mathrm{e}^{\sum \varphi^i \otimes \mathsf{F}_i} g^{-1} \mathrm{e}^{\sigma \otimes Y} = g' \mathrm{e}^{\sum (\varphi^i + \alpha_Y^i(g)\sigma) \otimes \mathsf{F}_i} g^{-1} \mathrm{e}^{\sum \beta_Y^j(g) \sigma \otimes \mathsf{E}_j} .$$

Proof. — The Lie algebra $\tilde{\mathfrak{g}} \equiv \tilde{\mathfrak{g}}(\Omega)$ inherits from $\mathfrak{g}$ a $\mathbb{Z}_2$-grading $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}^{\mathrm{even}} \oplus \tilde{\mathfrak{g}}^{\mathrm{odd}}$ by

$$\tilde{\mathfrak{g}}^{\mathrm{even}} = \bigoplus_{k \geq 0} \tilde{\mathfrak{g}}^{(2k)} , \quad \tilde{\mathfrak{g}}^{\mathrm{odd}} = \bigoplus_{k \geq 0} \tilde{\mathfrak{g}}^{(2k+1)} , \quad \tilde{\mathfrak{g}}^{(2k+\tau)} = \Omega^{2k+\tau} \otimes \mathfrak{g}_\tau \quad (\tau = 0, 1) .$$

Notice that $[\tilde{\mathfrak{g}}^{(k)}, \tilde{\mathfrak{g}}^{(l)}] \subset \tilde{\mathfrak{g}}^{(k+l)}$, so that $\oplus_{k \geq 1} \tilde{\mathfrak{g}}^{(k)} =: \mathfrak{n}$ is a nilpotent Lie subalgebra.

Proving the lemma essentially amounts to showing that, for any $g', g \in G$, there exist unique elements $\Xi_1 \in \tilde{\mathfrak{g}}^{\mathrm{odd}}$ and $\Xi_0 \in \tilde{\mathfrak{g}}^{\mathrm{even}} \cap \mathfrak{n}$ with the property that

$$g' \mathrm{e}^{\xi} g^{-1} \mathrm{e}^{\sigma \otimes Y} = g' \mathrm{e}^{\xi + \Xi_1} g^{-1} \mathrm{e}^{\Xi_0} .$$

For the purpose of establishing such an identity, we translate by $(g' \mathrm{e}^{\xi})^{-1}$ on the left and by g on the right, and then take the logarithm:

$$\sigma \otimes \mathrm{Ad}(g)^{-1} Y = \ln \left(\mathrm{e}^{-\xi} \mathrm{e}^{\xi + \Xi_1} \mathrm{e}^{\mathrm{Ad}(g)^{-1} \Xi_0} \right) .$$

This equation is equivalent to the previous one, as the exponential map from the nilpotent Lie algebra $\mathfrak{n} = \oplus_{k \geq 1} \tilde{\mathfrak{g}}^{(k)}$ to the nilpotent Lie group $\exp(\mathfrak{n})$ is a bijection.

It is clear that both unknowns Ξ_1 and Ξ_0 must contain at least one factor of the generator σ (and hence exactly one such factor, since $\sigma^2 = 0$). Using a standard formula from Lie theory we now combine the first two factors under the logarithm:

$$\mathrm{e}^{-\xi}\mathrm{e}^{\xi+\Xi_1} = 1 + T_\xi(\Xi_1) = \mathrm{e}^{T_\xi(\Xi_1)} , \quad T_\xi = \sum_{k\geq 0} \frac{\mathrm{ad}^k(-\xi)}{(k+1)!} .$$

Since $\mathrm{e}^{T_\xi(\Xi_1)}\mathrm{e}^{\mathrm{Ad}(g)^{-1}\Xi_0} = \mathrm{e}^{T_\xi(\Xi_1)+\mathrm{Ad}(g)^{-1}\Xi_0}$ by $\sigma^2 = 0$, our equation becomes

$$\sigma \otimes \mathrm{Ad}(g)^{-1} Y = T_\xi(\Xi_1) + \mathrm{Ad}(g)^{-1}\Xi_0 ,$$

in which form it can be solved for $\Xi_1 = \sum_{k\geq 1} \Xi_1^{(2k-1)}$ and $\Xi_0 = \sum_{k\geq 1} \Xi_0^{(2k)}$ by recursion in $k = 0,1,2,\ldots$. The first few terms of the solution are

$$\Xi_1^{(1)} = \sigma \otimes \mathrm{Ad}(g)^{-1} Y , \quad \Xi_0^{(2)} = \tfrac{1}{2!}\mathrm{Ad}(g)[\xi, \Xi_1^{(1)}] , \quad \Xi_1^{(3)} = -\tfrac{1}{3!}[\xi,[\xi,\Xi_1^{(1)}]] ,$$

and so on. The recursion terminates by nilpotency of the Lie algebra $\mathfrak{n}$.

All components $\Xi_\tau^{(2k-\tau)}$ of the solution are linear in Y and depend differentiably (in fact, analytically) on $g \in G$. By going back to the original equation $g'\mathrm{e}^{\xi}g^{-1}\mathrm{e}^{\sigma\otimes Y} = g'\mathrm{e}^{\xi+\Xi_1}g^{-1}\mathrm{e}^{\Xi_0}$, and expanding the solution for Ξ_1 and Ξ_0 in the chosen bases,

$$\Xi_1 = \sum \alpha_Y^i(g)\sigma \otimes \mathsf{F}_i , \quad \Xi_0 = \sum \beta_Y^j(g)\sigma\otimes \mathsf{E}_j ,$$

we arrive at the statement of the lemma. □

Proposition 3.2. — *The odd component $\mathfrak{g}_1 \subset \mathfrak{g}$ acts on sections $\Phi \in \mathscr{F}$ by $Y \mapsto \widehat{Y}^R$,*

$$(\widehat{Y}^R\Phi)(g',g) := \sum \alpha_Y^i(g)\frac{\partial}{\partial\varphi^i}\Phi(g',g) - \sum \beta_Y^j(g)\,(\widehat{\mathsf{E}}_j^R\Phi)(g',g) ,$$

where α^i and β^j are the functions of Lem. 3.1, the operator $\partial/\partial\varphi^i := \iota(\mathsf{F}_i)$ is the odd derivation of $\wedge(\mathfrak{g}_1^*)$ by contraction, and $\widehat{\mathsf{E}}_j^R$ is the left-invariant differential operator determined by $\mathsf{E}_j \in \mathfrak{g}_0$.

Proof. — We interpret the identity of Lem. 3.1 as saying that a right multiplication of the supermatrix $g'\mathrm{e}^{\xi}g^{-1}$ by $\mathrm{e}^{\sigma\otimes Y}$ is the same as shifting the generators $\varphi^i \to \varphi^i + \sigma\alpha_Y^i$ and making a right translation by $\mathrm{e}^{\sum\beta_Y^j\sigma\otimes\mathsf{E}_j}$. This observation motivates the proposed formula for $\widehat{Y}^R$, by linearization in the parameter σ (the sign in the second term of the expression for $\widehat{Y}^R$ stems from $\beta^j\sigma = -\sigma\beta^j$). It is easy to check that $\widehat{Y}^R$ really does map $\mathscr{F}$ into $\mathscr{F}$, i.e., if Φ is a G-equivariant map $\Phi : P \to \wedge(\mathfrak{g}_1^*)$, then so is $\widehat{Y}^R\Phi$.

Let now σ_1, σ_2 be *two* anti-commuting parameters, and let $Y_1, Y_2 \in \mathfrak{g}_1$. Then, since

$$\mathrm{e}^{\sigma_1\otimes Y_1}\mathrm{e}^{\sigma_2\otimes Y_2}\mathrm{e}^{-\sigma_1\otimes Y_1}\mathrm{e}^{-\sigma_2\otimes Y_2} = \mathrm{e}^{\sigma_2\sigma_1\otimes[Y_1,Y_2]} ,$$

it immediately follows that the correspondence $Y \mapsto \widehat{Y}^R$ preserves the Lie superbracket of $\mathfrak{g}$, i.e., one has the anti-commutation relations

$$[\widehat{Y}_1^R, \widehat{Y}_2^R] \equiv \widehat{Y}_1^R\widehat{Y}_2^R + \widehat{Y}_2^R\widehat{Y}_1^R = \widehat{[Y_1,Y_2]}^R .$$

(Please be advised that if we had used a Grassmann envelope of the first kind, as in (1.19), then there would have been a sign inconsistency in this equation.) Similarly, for $X \in \mathfrak{g}_0$ and $Y \in \mathfrak{g}_1$ one has $[\widehat{X}^R, \widehat{Y}^R] = \widehat{[X,Y]}^R$. $\square$

What we have described is the action on the right. But the same construction can be done on the left, and of course the resulting correspondence $Y \mapsto \widehat{Y}^L$ again is an odd derivation which preserves the bracket relations of $\mathfrak{g}$.

3.2.4. Summary. — Thus for every $Y \in \mathfrak{g}_1$ we are given two odd first-order differential operators $\widehat{Y}^L$ and $\widehat{Y}^R$ on sections $\Phi \in \mathscr{F}$ such that the superbracket relations

$$[\widehat{Y}_1^i, \widehat{Y}_2^i] = \widehat{[Y_1, Y_2]}^i , \quad [\widehat{X}^i, \widehat{Y}^i] = \widehat{[X,Y]}^i$$

hold for $i = L, R$ and all $X \in \mathfrak{g}_0$ and $Y, Y_1, Y_2 \in \mathfrak{g}_1$. The actions on the left and right commute in the graded-commutative sense.

Unlike the action of the Lie algebra $\mathfrak{g}_0$, that of the odd part $\mathfrak{g}_1$ does not readily exponentiate. We therefore leave this at the infinitesimal level.

Let us now summarize the educational material of this subsection.

Definition 3.3. — *Given a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, and a Lie group G (with Lie algebra $\mathfrak{g}_0$) acting on the vector space $\mathfrak{g}_1$ by the adjoint representation* Ad*, there is a principal bundle $P = G \times G \to G$ and an associated vector bundle $F \to G$ with total space $F = P \times_G \mathfrak{g}_1$. If $\mathfrak{g} = \mathfrak{gl}$ or $\mathfrak{g} = \mathfrak{osp}$, then by the Lie supergroup $\mathfrak{G} = (\mathfrak{g}, G)$ we mean the graded-commutative algebra of sections $\mathscr{F} = \Gamma(G, \wedge F^*)$ carrying the canonical actions of $\mathfrak{g}$ by left- or right-invariant derivations. A section of $\mathscr{F}$ is also referred to as a (super)function on $\mathfrak{G}$. The component in $\mathrm{C}^\infty(G)$ of a section $s \in \mathscr{F}$ is called the numerical part of s and is denoted by* $\mathrm{num}(s)$.

Remark. — Supermanifolds, of which Lie supergroups are special examples, are in principle constructed by using the language of sheaves, i.e., by joining together locally defined functions (here with values in an exterior algebra) by means of transition functions on overlapping domains. The transition functions for a general supermanifold respect only the $\mathbb{Z}_2$-grading (not the $\mathbb{Z}$-grading) of exterior algebras.

What is special about the supermanifolds defined above is that their structure sheaf *does* respect a $\mathbb{Z}$-grading, as it reduces to that of a vector bundle. Therefore it was not compulsory to use sheaf-theoretic language in formulating Def. 3.3. Nevertheless, we will sometimes refer to $\mathscr{F} = \Gamma(G, \wedge F^*)$ as a sheaf of graded-commutative algebras.

In the case of $\mathfrak{g} = \mathfrak{gl}_{p|q}$ and $G = \mathrm{GL}_p \times \mathrm{GL}_q$ one writes $\mathfrak{G} = \mathrm{GL}_{p|q}$ for short.

3.3. What's a representation in the supergroup setting?— If G is a Lie group (or, for that matter, any group), a representation of G is given by a $\mathbb{K}$-vector space V and a homomorphism from G into the group of $\mathbb{K}$-linear transformations of V, i.e., a mapping $\rho : G \to \mathrm{GL}(V)$ which respects the group multiplication law:

$$\rho(g_1 g_2) = \rho(g_1)\rho(g_2) .$$

In the Lie supergroup setting, since we have purposely avoided defining what is meant by the group multiplication law, we have to give meaning to the word ”representation” in an alternative (but equivalent) way.

Recall that the basic data of a Lie supergroup $\mathfrak{G}$ are a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and a Lie group G with $\mathrm{Lie}(G) = \mathfrak{g}_0$. To specify a representation of $\mathfrak{G}$, one first of all needs a representation space, which in the current context has to be a $\mathbb{Z}_2$-graded $\mathbb{K}$-vector space $V = V_0 \oplus V_1$. For present use recall from Sect. 1.2 the definition of the Lie superalgebra $\mathfrak{gl}(V) = \mathfrak{gl}(V)_0 \oplus \mathfrak{gl}(V)_1$, and let $e \in G$ be the neutral element.

Definition 3.4. — *A representation of a Lie supergroup $\mathfrak{G} = (\mathfrak{g}, G)$ on a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$ is given by a homomorphism ρ_* of Lie superalgebras*

$$\rho_* : \ \mathfrak{g}_0 \to \mathfrak{gl}(V)_0 \ , \quad \rho_* : \ \mathfrak{g}_1 \to \mathfrak{gl}(V)_1 \ ,$$

and a homomorphism of Lie groups

$$\rho : \ G \to \big(\mathrm{GL}(V_0) \times \mathrm{GL}(V_1) \hookrightarrow \mathrm{GL}(V)\big) \ ,$$

which are compatible in the sense that $(\mathrm{d}\rho)_e = \rho_*|_{\mathfrak{g}_0}$.

Remark. — Note that for the case of a finite-dimensional representation space V, compatibility immediately implies that

$$\rho(g)\,\rho_*(Y)\,\rho(g^{-1}) = \rho_*\big(\mathrm{Ad}(g)Y\big) \quad (g \in G\,,\ Y \in \mathfrak{g}_1)\ ,$$

by the formulas $\rho(\mathrm{e}^{tX}) = \mathrm{e}^{t\,(\mathrm{d}\rho)_e(X)}$ (for $t \in \mathbb{K}$) and $\mathrm{Ad} \circ \exp = \exp \circ \mathrm{ad}$.

3.4. What’s a character of G**?—** If we are given a representation (V, ρ_*, ρ) of the Lie supergroup $\mathfrak{G} = (\mathfrak{g}, G)$, we can say precisely – in the language of Sect. 3.1 – what is meant by its character χ. The precise meaning of χ comes from its definition as an element of $\mathscr{F} = \Gamma(G, \wedge F^*)$, i.e., as a section of the vector bundle

$$\wedge(F^*) = P \times_G \wedge(\mathfrak{g}_1^*) \ \to \ G \ .$$

This section χ is constructed from the representation (V, ρ_*, ρ) as follows.

As before, fix some basis $\{\mathsf{F}_i\}$ of $\mathfrak{g}_1$, denote the dual basis of $\mathfrak{g}_1^*$ by $\{\varphi^i\}$, and recall the tautological object $\xi = \sum \varphi^i \otimes \mathsf{F}_i$. Then, fixing some pair $(h', h) \in G \times G = P$, consider the task of representing the formal object $\Xi = h'\,\mathrm{e}^{\xi} h^{-1}$ as an operator on V, with coefficients in $\wedge(\mathfrak{g}_1^*)$. The obvious choice of operator is

$$\rho(h')\,\mathrm{e}^{\sum \varphi^i \rho_*(\mathsf{F}_i)} \rho(h^{-1})\ ,$$

which is to be viewed as an element of the Grassmann envelope $\mathrm{End}_{\wedge \mathfrak{g}_1^*}(V)$.

Since $\mathrm{End}_{\wedge \mathfrak{g}_1^*}(V)$ is of the second kind, the corresponding supertrace is defined by

$$\mathrm{STr}\,(\alpha \otimes X)(\beta \otimes Y) := \beta\alpha\,\mathrm{STr}_V(XY)\ .$$

For later use, note that the supertrace on a Grassmann envelope of first or second kind always has the cyclic property: $\mathrm{STr}\,(\Xi\Theta) = \mathrm{STr}\,(\Theta\Xi)$.

Now take the supertrace of the operator above to define

$$\begin{aligned}\chi(h',h) &:= \mathrm{STr}_V\,\rho(h')\,\mathrm{e}^{\sum\varphi^i\rho_*(\mathsf{F}_i)}\rho(h^{-1})\\ &= \mathrm{STr}_V\,\rho(h'h^{-1})+\sum\varphi^i\,\mathrm{STr}_V\,\rho(h')\rho_*(\mathsf{F}_i)\rho(h^{-1})\\ &+ \tfrac{1}{2}\sum\varphi^j\varphi^i\,\mathrm{STr}_V\,\rho(h')\rho_*(\mathsf{F}_i)\rho_*(\mathsf{F}_j)\rho(h^{-1})+\dots\,,\end{aligned}$$

where the Taylor expansion of the exponential function terminates at finite order since the φ^i are nilpotent. If the dimension of V is finite, this definition makes perfect sense. In the infinite-dimensional case, the convergence of $\mathrm{STr}_V\,\rho(h'h^{-1})$ etc. is an issue and $\chi(h',h)$ will typically exist only for (h',h) in a suitable subset of $G\times G$; for simplicity of the exposition, we here assume this complication to be absent.

When h and h' are allowed to vary, χ becomes a function on $G\times G=P$ which takes values in $\wedge(\mathfrak{g}_1^*)$ and is equivariant with respect to G. Indeed,

$$\chi(h'g,hg)=\mathrm{STr}_V\,\rho(h')\,\mathrm{e}^{\sum\varphi^i\rho(g)\rho_*(\mathsf{F}_i)\rho(g^{-1})}\rho(h^{-1})\,.$$

By the compatibility of ρ and ρ_*,

$$\sum\varphi^i\rho(g)\rho_*(\mathsf{F}_i)\rho(g^{-1})=\sum\varphi^i\rho_*\big(\mathsf{F}_j\,\mathrm{Ad}(g)^j{}_i\big)=\sum\varphi^i\,\mathrm{Ad}^*(g)_i{}^j\rho_*(\mathsf{F}_j)\,,$$

and therefore

$$\chi(h',h)=\mathrm{Ad}^*(g^{-1})\,\chi(h'g,hg)\,,$$

where $g\mapsto\mathrm{Ad}^*(g^{-1})$ is the induced G-representation on $\wedge(\mathfrak{g}_1^*)$. Hence χ is an element of $\mathrm{C}^\infty(P,\wedge\mathfrak{g}_1^*)^G$ or, equivalently, a section of $\Gamma(G,\wedge F^*)$.

Definition 3.5. — *The character determined by the representation (V,ρ_*,ρ) of a Lie supergroup $\mathfrak{G}=(\mathfrak{g},G)$ is defined to be the section $\chi\in\Gamma(G,\wedge F^*)$ given by*

$$\chi(gh^{-1})=\big[(g,h)\,;\mathrm{STr}_V\,\rho(g)\,\mathrm{e}^{\sum\varphi^i\rho_*(\mathsf{F}_i)}\rho(h^{-1})\big]\,,$$

whenever this exists. We also write $\Xi=g\,\mathrm{e}^{\xi}h^{-1}$ and $\chi(\Xi)=\mathrm{STr}_V\,\rho(\Xi)$ for short.

Remark. — The numerical part $\mathrm{num}(\chi)\in\Gamma(G,\wedge^0F^*)$ coincides with the character of the $\mathbb{Z}_2$-graded G-representation $(V_0\oplus V_1,\rho)$ in the $\mathbb{Z}_2$-graded sense:

$$\mathrm{num}(\chi)(x)=\mathrm{Tr}_{V_0}\,\rho(x)-\mathrm{Tr}_{V_1}\,\rho(x)\,.$$

3.5. χ is an eigenfunction of all Laplace-Casimir operators. — Primitive characters, i.e., characters of irreducible representations, are special functions with special properties. Foremost among these is their being joint eigenfunctions of the ring of invariant differential operators. Let us review this general property in the superalgebra setting, which is where it will be exploited below. In this subsection we put $\mathbb{K}=\mathbb{C}$.

If $\mathfrak{g}$ is a Lie superalgebra, the universal enveloping algebra $\mathsf{U}(\mathfrak{g})$ is the associative algebra generated by $\mathfrak{g}$ with the bracket relations of $\mathfrak{g}$ being understood. A *Casimir invariant* of $\mathfrak{g}$ then is an element I in the center of $\mathsf{U}(\mathfrak{g})$, i.e, a polynomial I in the generators of $\mathfrak{g}$ with the property $[I,X]=0$ for all $X\in\mathfrak{g}$. For example, if $E_1,\dots,E_d$

(with $d = \dim \mathfrak{g}$) is a basis of $\mathfrak{g}$ and $Q : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is a non-degenerate $\mathfrak{g}$-invariant quadratic form, the quadratic Casimir invariant is

$$I_2 = \sum Q^{ij} E_i E_j \ ,$$

where the coefficients Q^{ij} are determined by $\sum_j Q^{ij} Q_{jk} = \delta^i_k$ and $Q_{ij} = Q(E_i, E_j)$.

To describe the Casimir invariants of the Lie superalgebra $\mathfrak{g} = \mathfrak{gl}_{p|q}$, let $E_{ij}^{\sigma\tau}$ with $\sigma, \tau = 0, 1$ be a standard basis of $\mathrm{End}(\mathbb{C}^{p|q})$. This means that the parity of $E_{ij}^{\sigma\tau}$ is $\sigma + \tau$ (mod 2), the range of a lower index is understood to be $i \in \{1, 2, \ldots, p\}$ if the corresponding upper index is $\sigma = 1$ and $i \in \{1, 2, \ldots, q\}$ if $\sigma = 0$, and one has

$$E_{ij}^{\sigma\tau} E_{i'j'}^{\sigma'\tau'} = \delta^{\tau\sigma'} \delta_{ji'} E_{ij'}^{\sigma\tau'} \ .$$

***Lemma 3.6* (Berezin)**. — *There exists a degree-ℓ Casimir invariant I_ℓ of $\mathfrak{gl}_{p|q}$ for every $\ell \in \mathbb{N}$. Its expression in terms of a standard basis of generators $E_{ij}^{\sigma\tau}$ is*

$$I_\ell = \sum E_{j_1 j_2}^{\sigma_1\sigma_2} (-1)^{\sigma_2} E_{j_2 j_3}^{\sigma_2\sigma_3} (-1)^{\sigma_3} \cdots E_{j_{\ell-1} j_\ell}^{\sigma_{\ell-1}\sigma_\ell} (-1)^{\sigma_\ell} E_{j_\ell j_1}^{\sigma_\ell \sigma_1} \ .$$

Remark. — This formula for I_ℓ is Eqs. (4.35-37) of [**2**]. From the basic brackets

$$[E_{ij}^{\sigma\tau}, E_{i'j'}^{\sigma'\tau'}] = E_{ij'}^{\sigma\tau'} \delta_{ji'} \delta^{\tau\sigma'} - (-1)^{(\sigma+\tau)(\sigma'+\tau')} E_{i'j}^{\sigma'\tau} \delta_{j'i} \delta^{\tau'\sigma} \ , \tag{3.1}$$

it is in fact easy to check that $[X, I_\ell] = 0$ for all $X \in \mathfrak{gl}_{p|q}$.

Now turn to the functions on a Lie supergroup $\mathfrak{G} = (\mathfrak{g}, G)$, i.e., to the sections Φ of the algebra $\mathscr{F} = \Gamma(G, \wedge F^*)$. Every Casimir invariant $I \in \mathsf{U}(\mathfrak{g})$ determines an invariant differential operator $D(I)$ – also called a Laplace-Casimir operator – on such functions. In the case of the quadratic Casimir invariant I_2 this is

$$D(I_2)\Phi = \sum Q^{ij} \widehat{E}_i^L \widehat{E}_j^L \Phi = \sum Q^{ij} \widehat{E}_i^R \widehat{E}_j^R \Phi \ ,$$

where $E_i \mapsto \widehat{E}_i^{L,R}$ are the canonical right- and left-invariant actions of the Lie superalgebra $\mathfrak{g}$ on the Lie supergroup $\mathfrak{G}$ (see Def. 3.3). More generally, $D(I)$ is obtained by replacing in the polynomial expression for I each generator of $\mathfrak{g}$ by the corresponding right-invariant or left-invariant differential operator.

Assume now an irreducible representation (V, ρ_*, ρ) of the Lie supergroup $\mathfrak{G}$ to be given. Applying any Laplace-Casimir operator $D(I)$ to the character $\chi(\Xi)$ one gets

$$\big(D(I)\chi\big)(\Xi) = \mathrm{STr}_V \, \rho_*(I) \rho(\Xi) \ ,$$

where ρ_* has been extended from $\mathfrak{g}$ to $\mathsf{U}(\mathfrak{g})$. Recall that we now are working over $\mathbb{K} = \mathbb{C}$. Since the representation ρ_* is irreducible and the Casimir invariant I commutes with all generators of $\mathfrak{g}$, it follows that the operator $\rho_*(I)$ is a multiple of unity by Schur's lemma: $\rho_*(I) = \lambda(I) \, \mathrm{Id}_V$ with $\lambda(I) \in \mathbb{C}$. Thus χ is an eigenfunction:

$$D(I)\chi = \lambda(I)\, \chi \ .$$

This fact will be exploited in Sect. 4.6.

3.6. Radial functions. — An important fact about the characters of a group G is that they are constant on conjugacy classes: $\chi(g) = \chi(hgh^{-1})$ for $g, h \in G$. Functions $f : G \to \mathbb{C}$ with this property are called *radial*. In the case of a Lie group G the infinitesimal version of the radial property is

$$0 = \frac{d}{ds} f(\mathrm{e}^{-sX} g\, \mathrm{e}^{sX})\Big|_{s=0} = \big(\widehat{X}^L + \widehat{X}^R\big) f(g) \, .$$

Notice that the notation used here is consistent with our earlier definition of the actions on the left and right, $\widehat{X}^{L,R}$, by the identification $f_2(g,h) \equiv f_1(gh^{-1})$.

Definition 3.7. — *A section $\Phi \in \mathscr{F}$ of a Lie supergroup $\mathfrak{G} = (\mathfrak{g}, G)$ is called radial if $\Phi(gxg^{-1}) = \Phi(x)$ for all $g, x \in G$, and if*

$$\big(\widehat{Y}^L + \widehat{Y}^R\big)\Phi = 0$$

holds for all $Y \in \mathfrak{g}_1$, the odd part of the Lie superalgebra $\mathfrak{g}$.

Remark. — An important example of a radial section for the case of $\mathfrak{g} = \mathfrak{gl}(V_1 \oplus V_0)$ and $G = \mathrm{GL}(V_1) \times \mathrm{GL}(V_0)$ is the superdeterminant (see Def. 1.5).

The material reviewed in Sects. 3.1-3.4 did not depend on the choice of number field $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$. However, to do analysis with radial functions we now wish to abandon the basic case of a complex Lie superalgebra $\mathfrak{g}$ with complex Lie group G and restrict the discussion to a *real* subgroup $G_\mathbb{R} \subset G$.

Hence, let $G_\mathbb{R} \equiv K$ be a connected compact Lie group. The set of conjugacy classes then is parameterized by the elements t in a Cartan subgroup, or maximal torus, $T \subset K$. It follows that a radial function Φ on K determines a function $t \mapsto f(t)$ by restriction, $f := \Phi\big|_T$. Conversely, a function f on T is known to extend to a radial function Φ on K provided that f is invariant under the action of the so-called Weyl group W of K.

Turning to the case of a complex Lie supergroup $\mathfrak{G} = (\mathfrak{g}, G)$, let $K \subset G$ be a connected compact Lie group whose Lie algebra is a compact real form of $\mathfrak{g}_0 = \mathrm{Lie}(G)$. In the case of $\mathfrak{G} = \mathrm{GL}_{m|n}$ for example, take $K = \mathrm{U}_m \times \mathrm{U}_n$. Then let $\mathscr{F}'$ denote the sheaf of algebras $\mathscr{F}$ restricted to K. Note that the restriction $\mathscr{F}'$ still carries the action of the complex Lie superalgebra $\mathfrak{g}$ by left-invariant and right-invariant differential operators.

Now fix some maximal torus T of K. A radial section $\Phi \in \mathscr{F}'$ determines a function $f : T \to \mathbb{C}$ by restriction and truncation to the numerical part,

$$f := \mathrm{num}(\Phi)\big|_T \, ,$$

and this function f again is invariant under the Weyl group W of K. (Please be advised that the converse is no longer true! In order for a W-invariant function f on T to extend to a radial section $\Phi \in \mathscr{F}'$ some extra regularity conditions must be fulfilled; see Part II, Chapter 3, Thm. 3.1 of [**2**]. However, in the application we are aiming at, namely the character χ of Sect. 2, extendability will be automatic by construction. Hence there is no need to discuss the full set of conditions for extendability here.)

For every Casimir invariant $I \in \mathsf{U}(\mathfrak{g})$ we have a Laplace-Casimir operator $D(I)$ on $\mathscr{F}$. If $\Phi \in \mathscr{F}$ is radial, i.e., $\Phi = L_g R_g \Phi$ for all $g \in G$ and $(\widehat{Y}^L + \widehat{Y}^R)\Phi = 0$ for all $Y \in \mathfrak{g}_1$, then so is $D(I)\Phi$, since $D(I)$ commutes with all of the L_g and R_g and $\widehat{Y}^{L,R}$.

Thus the subalgebra of radial sections in $\mathscr{F}$ is invariant under the application of $D(I)$. The restriction $\dot{D}(I)$ of $D(I)$ to this invariant subalgebra is called the *radial part*.

Given a compact real form $K = G_{\mathbb{R}} \subset G$, one can transfer this discussion to the subalgebra of radial sections of the restriction $\mathscr{F}' = \mathscr{F}|_K$, which is still invariant under the application of the Laplace-Casimir operators $D(I)$. The radial parts $\dot{D}(I)$ for $\mathfrak{G} = \mathrm{GL}$ or $\mathfrak{G} = \mathrm{OSp}$ can then be described as differential operators on the algebra of extendable W-invariant functions on a maximal torus $T \subset K$. We will write them down for the case of $\mathfrak{G} = \mathrm{GL}_{m|n}$ with $K = \mathrm{U}_m \times \mathrm{U}_n$ in Sect. 3.8.

3.7. Block diagonalization of supermatrices. — Berezin has given a description [**2**] of the radial parts of the Laplace-Casimir operators for the cases of $\mathfrak{g} = \mathfrak{gl}$ and $\mathfrak{g} = \mathfrak{osp}$. Since our paper will make heavy use of it, we now give a flavor of the first step of that theory. (For this first step, the choice of $K \subset G$ is not yet relevant.)

Recalling the material of Sect. 3.2.2, let $\tilde{\mathfrak{g}}(\Omega)$ be the Grassmann envelope (of first or second kind) of the Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ by the Grassmann algebra $\Omega = \wedge(\mathfrak{g}_1^*)$. The Lie algebra $\tilde{\mathfrak{g}} \equiv \tilde{\mathfrak{g}}(\Omega)$ carries a $\mathbb{Z}_2$-grading $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}^{\mathrm{even}} \oplus \tilde{\mathfrak{g}}^{\mathrm{odd}}$ and also a compatible $\mathbb{Z}$-grading $\tilde{\mathfrak{g}} = \oplus_{k\geq 0}\, \tilde{\mathfrak{g}}^{(k)}$. From $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$ one has an induced adjoint action of G on $\tilde{\mathfrak{g}}$ and each of its subspaces $\tilde{\mathfrak{g}}^{(k)}$ for $k \geq 0$ by $\alpha \otimes X \mapsto \alpha \otimes \mathrm{Ad}(g) X$.

Proposition 3.8. — *Let $g \in G$ be regular in the sense that* $\mathrm{Ad}(g) - \mathrm{Id}$ *has zero kernel as a linear operator on $\mathfrak{g}_1$. Then for any $\Theta_1 \in \tilde{\mathfrak{g}}^{\mathrm{odd}}$ the supermatrix $g\,\mathrm{e}^{\Theta_1}$ can be block diagonalized by nilpotents, i.e., there exist unique elements $\Xi_1^{(2k-1)} \in \tilde{\mathfrak{g}}^{(2k-1)}$ and $\Xi_0^{(2k)} \in \tilde{\mathfrak{g}}^{(2k)}$ for $k \geq 1$ such that*

$$g\,\mathrm{e}^{\Theta_1} = \mathrm{e}^{\Xi_1} g\, \mathrm{e}^{\Xi_0} \mathrm{e}^{-\Xi_1}$$

holds with $\Xi_1 = \sum_{k\geq 1} \Xi_1^{(2k-1)} \in \tilde{\mathfrak{g}}^{\mathrm{odd}}$ and $\Xi_0 = \sum_{k\geq 1} \Xi_0^{(2k)} \in \tilde{\mathfrak{g}}^{\mathrm{even}}$.

Proof. — Translate by g^{-1} on the left and take the logarithm:

$$\Theta_1 = \ln\left(\mathrm{e}^{\mathrm{Ad}(g^{-1})\Xi_1}\, \mathrm{e}^{\Xi_0} \mathrm{e}^{-\Xi_1}\right) .$$

This equation is equivalent to the previous one, as the exponential map from the nilpotent Lie algebra $\mathfrak{n} = \oplus_{k\geq 1} \tilde{\mathfrak{g}}^{(k)}$ to the nilpotent Lie group $\exp(\mathfrak{n})$ is a bijection. Using the Baker-Campbell-Hausdorff formula to compute the logarithm, one gets an expansion

$$\Theta_1 = \mathrm{Ad}(g^{-1})\Xi_1 - \Xi_1 + \Xi_0 - \tfrac{1}{2}[\Xi_0, \Xi_1] + \tfrac{1}{2}[\mathrm{Ad}(g^{-1})\Xi_1, \Xi_0 - \Xi_1] + \ldots ,$$

which terminates at finite order by nilpotency. The equation in this form can be solved for the unknowns Ξ_1 and Ξ_0 by recursion in the $\mathbb{Z}$-degree. Indeed, expanding $\Theta_1 = \sum_{k\geq 1} \Theta_1^{(2k-1)}$ the degree-1 equation is

$$\Theta_1^{(1)} = \left(\mathrm{Ad}(g^{-1}) - \mathrm{Id}\right) \Xi_1^{(1)} ,$$

which has a unique solution for $\Xi_1^{(1)}$ because $\mathrm{Ad}(g^{-1}) - \mathrm{Id} = \mathrm{Ad}(g^{-1})(\mathrm{Id} - \mathrm{Ad}(g))$ is invertible by assumption. The degree-2 equation determines $\Xi_0^{(2)}$ from $\Xi_1^{(1)}$:

$$0 = \Xi_0^{(2)} - \tfrac{1}{2}[\mathrm{Ad}(g^{-1})\Xi_1^{(1)}, \Xi_1^{(1)}] .$$

The degree-3 equation determines $\Xi_1^{(3)}$ from $\Theta_1^{(3)}$ and known quantities of lower order, the degree-4 equation determines $\Xi_0^{(4)}$, and so on. $\square$

3.8. Radial parts of the Laplacians for $\mathrm{U}_{m|n}$. — We finish our exposition of background material by writing down the radial parts of the Laplace-Casimir operators for the case of $\mathfrak{G} = \mathrm{GL}_{m|n}$ with $K = \mathrm{U}_m \times \mathrm{U}_n$. This case is referred to as the *unitary Lie supergroup* $\mathrm{U}_{m|n}$. The Casimir invariants I_ℓ of $\mathrm{U}(\mathfrak{gl}_{m|n})$ were given in Lem. 3.6.

Fixing some choice of standard basis $E_{ii'}^{\sigma\sigma'}$ of $\mathrm{End}(\mathbb{C}^{m|n})$, let $T \simeq \mathrm{U}_1^{m+n}$ be the corresponding maximal torus of diagonal transformations t,

$$t = \sum\nolimits_{j=1}^{m} \mathrm{e}^{\mathrm{i}\psi_j} E_{jj}^{11} + \sum\nolimits_{l=1}^{n} \mathrm{e}^{\mathrm{i}\phi_l} E_{ll}^{00}$$

where the angular variables ψ_j and ϕ_l are real-valued local coordinates for T, and denote by D_k the degree-k radial differential operator

$$D_k = \sum_{j=1}^{m} \frac{\partial^k}{\partial \psi_j^k} - (-1)^k \sum_{l=1}^{n} \frac{\partial^k}{\partial \phi_l^k} .$$

***Theorem 3.9* (Berezin). —** *If J is the function defined (on a dense open set in T) by*

$$J = \frac{\prod_{1\le j<j'\le m} \sin^2\left(\frac{1}{2}(\psi_j - \psi_{j'})\right) \prod_{1\le l<l'\le n} \sin^2\left(\frac{1}{2}(\phi_l - \phi_{l'})\right)}{\prod_{j=1}^{m}\prod_{l=1}^{n} \sin^2\left(\frac{1}{2}(\psi_j - \phi_l)\right)} ,$$

the degree-ℓ Laplace-Casimir operator $D(I_\ell)$ for $\mathrm{U}_{m|n}$ has the radial part

$$\dot{D}(I_\ell) = J^{-1/2} P_\ell \circ J^{1/2} ,$$

where $P_\ell = D_\ell + Q_{\ell-1}$ and $Q_{\ell-1}$ is some polynomial in the homogeneous operators D_k of total degree less than ℓ.

***Remark*. —** The expression for the highest-order term, $\dot{D}(I_\ell) = J^{-1/2} D_\ell \circ J^{1/2} + \ldots$, is from Thm. 3.2, Eq. (3.47), of [**2**], specialized to the case of $\mathrm{U}_{m|n}$. The lower-order terms $Q_{\ell-1}$ are known to be polynomial in the D_k by Thm. 4.4 of [**2**].

The square root $J^{1/2}$ does not exist as a single-valued function on U_1^{m+n} for every dimension $m|n$, but the square root ambiguity cancels in the differential operator $\dot{D}(I_\ell)$.

In the special case where dimensions match, one has the following simplification.

***Lemma 3.10* (Berezin). —** *For $m = n$ the function J can be put in the form $J =$ $\mathrm{Det}^2(A)$ where A is the $n \times n$ matrix with entries*

$$A_{jl} = \frac{1}{\sin\left(\frac{1}{2}(\psi_j - \phi_l)\right)} .$$

Proof. — In the Cauchy determinant formula

$$\frac{\prod_{j<l}(x_j - x_l)(y_l - y_j)}{\prod_{j,l}(x_j - y_l)} = \mathrm{Det}\left(\frac{1}{x_j - y_l}\right)_{j,l=1,\ldots,n} ,$$

make the substitution $x_j = \mathrm{e}^{\mathrm{i}\psi_j}$ and $y_l = \mathrm{e}^{\mathrm{i}\phi_l}$. The statement then immediately follows from Euler's formula $2\mathrm{i}\sin z = \mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}$ on dividing both sides by a suitable factor. $\square$

Corollary 3.11. — *For $m = n$ the constant term of the lower-order differential operator $Q_{\ell-1}(\partial/\partial\psi_1, \ldots, \partial/\partial\psi_n, \partial/\partial\phi_1, \ldots, \partial/\partial\phi_n)$ in the radial part $\dot{D}(I_\ell)$ vanishes for all $\ell \in \mathbb{N}$: $Q_{\ell-1}(\mathbf{0},\mathbf{0}) = 0$.*

Proof. — The first step is to show that for $m = n$ the square root $J^{1/2}$ is annihilated by each of the homogeneous radial differential operators D_k. For that purpose, write

$$J^{1/2} := \mathrm{Det}(A) = \mathrm{Det}\left(\frac{2\mathrm{i}\,\mathrm{e}^{-\frac{\mathrm{i}}{2}(\psi_j - \phi_l)}}{1 - \mathrm{e}^{-\mathrm{i}(\psi_j - \phi_l)}}\right)_{j,l=1,\ldots,n} .$$

Now express the determinant as a sum over permutations, move the ϕ_l into the upper half of the complex plane, and expand the factors $(1 - \mathrm{e}^{-\mathrm{i}(\psi_j - \phi_l)})^{-1}$ as geometric series. The result is an absolutely convergent series, each term of which is readily seen to be annihilated by every one of the D_k. Thus $D_k J^{1/2} = 0$ for all $k \in \mathbb{N}$.

Now, since $Q_{\ell-1}$ is a polynomial in the operators D_k, it follows that when $Q_{\ell-1}$ is applied to $J^{1/2}$, only the zero-order term in $Q_{\ell-1}$ can give a nonzero answer: $Q_{\ell-1}J^{1/2} = J^{1/2}Q_{\ell-1}(\mathbf{0},\mathbf{0})$. But the differential operator $\dot{D}(I_\ell)$ annihilates the constants; thus one has $\dot{D}(I_\ell) \cdot 1 = 0$ and Thm. 3.9 implies $0 = J^{-1/2}(D_\ell + Q_{\ell-1})J^{1/2} = Q_{\ell-1}(\mathbf{0},\mathbf{0})$. $\square$

Finding the radial part of an invariant differential operator is an algebraic problem (as opposed to an analytical one). It should therefore be clear that the formulas given in Thm. 3.9 hold not only for the case of $K = \mathrm{U}_m \times \mathrm{U}_n$ with maximal torus U_1^{m+n}, but admit analytic continuation to other real-analytic domains. It is in such a domain, M (with another maximal torus T), that we will use them to prove Thm. 1.1.

4. Determination of the character χ

In Sect. 2.6 we considered a $\mathbb{Z}_2$-graded complex vector space $V = U \otimes \mathbb{C}^N$ where

$$\begin{aligned} V &= V_1 \oplus V_0 = (V_1^+ \oplus V_1^-) \oplus (V_0^+ \oplus V_0^-) , \\ U &= U_1 \oplus U_0 = (U_1^+ \oplus U_1^-) \oplus (U_0^+ \oplus U_0^-) , \\ V_\tau^\pm &= U_\tau^\pm \otimes \mathbb{C}^N , \quad U_1^+ = U_0^+ = \mathbb{C}^p , \quad U_1^- = U_0^- = \mathbb{C}^q . \end{aligned}$$

In this tensor-product situation there is a Howe dual pair of a Lie group U_N and a Lie superalgebra $\mathfrak{gl}(U) = \mathfrak{gl}_{n|n}$ (with $n = p + q$) acting on the spinor-oscillator module

$$\mathscr{A}_V \simeq \wedge(V_1^+ \oplus V_1^{-*}) \otimes \mathrm{S}(V_0^+ \oplus V_0^{-*}) .$$

Prop. 2.10 states that the subspace $\mathscr{A}_V^{\mathrm{U}_N}$ of U_N-invariants in $\mathscr{A}_V$ is an irreducible representation space for $\mathfrak{gl}(U)$ and that, moreover, the autocorrelation function of ratios (1.5) coincides with the character associated with that representation.

In the current section we use the simplified notation

$$\mathscr{V}_\lambda \equiv \mathscr{A}_V^{\mathrm{U}_N} . \tag{4.1}$$

This notation expresses the two properties of $\mathscr{V}_\lambda$ being (i) a graded-commutative algebra built from V and (ii) an irreducible $\mathfrak{gl}(U)$-module with highest weight $\lambda \equiv \lambda_N$.

Recalling the definition of the superdeterminant, and specializing it to the case of $V = U \otimes \mathbb{C}^N$, we can cast the equality of expressions of Prop. 2.10 in the following form, where $m \otimes u$ is viewed as an even element of $\mathrm{End}(V) \simeq \mathrm{End}(U) \otimes \mathrm{End}(\mathbb{C}^N)$.

Corollary 4.1. —

$$\chi(m) = \mathrm{STr}_{\mathscr{V}_\lambda}\, \rho(m) = \int_{\mathrm{U}_N} \mathrm{SDet}(\mathrm{Id}_V - m \otimes u)^{-1}\, du\,.$$

All of our considerations of χ in Sect. 2 were restricted to the diagonal transformations $m = t$; it is, in fact, only the values on the diagonal that we ultimately want to know. Nevertheless, in order to establish good analytical control and actually compute these values, our next goal is to extend the function $t \mapsto \chi(t)$ to a section of the Lie supergroup $\mathfrak{G} = \mathrm{GL}_{n|n}$. We wish to do this by following Sect. 3.4, where we explained how to construct a $\mathfrak{G}$-representation from a representation ρ_* of the Lie superalgebra and a compatible representation ρ of the Lie group G underlying $\mathfrak{G}$.

In this quest, however, we are facing a difficulty: since the representation space $\mathscr{V}_\lambda$ is infinite-dimensional and some of the Lie algebra elements are represented by unbounded operators, the representation ρ_* does not exponentiate to all of the complex Lie group of linear transformations of the $\mathbb{Z}_2$-graded complex vector space U. We therefore require a more elaborate variant of the construction given in Sect. 3.4.

4.1. The good real structure. — Our first step toward extending the character function $t \mapsto \chi(t)$ is to identify a suitable real structure inside the complex Lie supergroup $\mathrm{GL}_{n|n}$ of the $\mathbb{Z}_2$-graded vector space $U \simeq \mathbb{C}^{n|n}$. Let $\mathrm{GL}(U_1) \times \mathrm{GL}(U_0) =: G$ be the group underlying the Lie supergroup GL of U. As should be clear from Prop. 2.3 (making a statement about semigroup elements), this complex Lie group is not the appropriate space on which to consider our character χ. Rather, the fact that $\dim \mathscr{V}_\lambda = \infty$ forces us to get a certain real subgroup $G_\mathbb{R} \subset G$ into play.

To introduce $G_\mathbb{R}$, let the decomposition $U_0 = U_0^+ \oplus U_0^-$ be encoded in an involution $s : U_0 \to U_0$ which has $U_0^+ = \mathbb{C}^p$ and $U_0^- = \mathbb{C}^q$ for its two eigenspaces:

$$s\,(u_+ + u_-) = u_+ - u_- \quad (u_\pm \in U_0^\pm)\,, \tag{4.2}$$

and let $U_0^s \equiv (U_0\,, s)$ mean U_0 equipped with the pseudo-unitary structure

$$U_0 \times U_0 \to \mathbb{C}\,, \quad (u, v) \mapsto \langle u, sv \rangle\,,$$

where $\langle\,,\rangle$ is the Hermitian scalar product of the Hermitian vector space $U_0 = \mathbb{C}^{p+q}$. The group of isometries of U_0^s is the non-compact Lie group

$$\mathrm{U}_{p,q} := \{g \in \mathrm{GL}(U_0) \mid g^\dagger s g = s\}\,. \tag{4.3}$$

Also, let U_n denote the isometry group of the Hermitian vector space $U_1 = \mathbb{C}^n$. For our purposes, the "good" real subgroup of G will turn out to be

$$G_\mathbb{R} = \mathrm{U}_n \times \mathrm{U}_{p,q}\,. \tag{4.4}$$

4.1.1. $G_{\mathbb{R}}$-principal bundle. — We now describe the $G_{\mathbb{R}}$-principal bundle which is to assume the role formerly played by $G \times G \to G$ in Sect. 3.1. Beginning with the bosonic sector, let $\mathrm{H}_{p,q}$ be the set

$$\mathrm{H}_{p,q} := \{g \in \mathrm{GL}(U_0) \mid g^\dagger s g < s\} \; . \tag{4.5}$$

If two group elements $g, h \in \mathrm{GL}(U_0)$ satisfy $g^\dagger s g < s$ and $h^\dagger s h < s$, then

$$(gh)^\dagger s(gh) = h^\dagger (g^\dagger s g) h < h^\dagger s h < s \; .$$

Thus $\mathrm{H}_{p,q}$ is a semigroup w.r.t. multiplication by composition, $(g,h) \mapsto gh$. Note also that $\mathrm{H}_{p,q}$ contains the pseudo-unitary group $\mathrm{U}_{p,q}$ in its closure $\overline{\mathrm{H}}_{p,q}$.

Now define an involutory automorphism $\sigma : \mathrm{GL}(U_0) \to \mathrm{GL}(U_0)$ by

$$\sigma(g) = s {g^{-1}}^\dagger s \; . \tag{4.6}$$

The set of σ-fixed points in $\mathrm{GL}(U_0)$ is the pseudo-unitary group $\mathrm{U}_{p,q}$.

Since the inequality $x^\dagger s x < s$ is invariant under left and right translations of x by the elements of $\mathrm{U}_{p,q}$, the semigroup $\mathrm{H}_{p,q}$ is a left and right $\mathrm{U}_{p,q}$-space. In Cor. 5.3 we will prove that $\mathrm{H}_{p,q}$ has the following closure property: if $x \in \mathrm{H}_{p,q}$ then $\sigma(x)^{-1} \in \mathrm{H}_{p,q}$.

Next, define a $\mathrm{U}_{p,q}$-principal bundle $P_0 \to M_0$ as follows. Let P_0 be the graph of the mapping $\sigma : \mathrm{H}_{p,q} \to \mathrm{GL}(U_0)$:

$$P_0 = \{(x,y) \in \mathrm{H}_{p,q} \times \mathrm{GL}(U_0) \mid y = \sigma(x)\} \; , \tag{4.7}$$

and take $M_0 \subset \mathrm{GL}(U_0)$ to be the set

$$M_0 = \{x\,\sigma(x)^{-1} \mid x \in \mathrm{H}_{p,q}\} \; . \tag{4.8}$$

Note $M_0 \subset \mathrm{H}_{p,q}$ from $\sigma(x)^{-1} \in \mathrm{H}_{p,q}$ and the fact that $\mathrm{H}_{p,q}$ is a semigroup. Since $\mathrm{U}_{p,q} \subset \mathrm{GL}(U_0)$ is the subgroup of fixed points of the automorphism σ, the map

$$P_0 \to M_0 \simeq P_0/\mathrm{U}_{p,q} \; , \quad (x, \sigma(x)) \mapsto x\,\sigma(x)^{-1} \; ,$$

defines a $\mathrm{U}_{p,q}$-principal bundle by the right $\mathrm{U}_{p,q}$-action $(x, \sigma(x)) \mapsto (xg, \sigma(xg))$.

The corresponding construction for the fermionic sector built on $U_1 = \mathbb{C}^n$ is simpler: let a U_n-principal bundle $P_1 \to M_1$ be defined by the projection map

$$P_1 := \mathrm{U}_n \times \mathrm{U}_n \to \mathrm{U}_n =: M_1 \; , \quad (x,y) \mapsto xy^{-1} \; . \tag{4.9}$$

The base manifold M_1 is a real-analytic submanifold of $\mathrm{GL}(U_1) = \mathrm{GL}(\mathbb{C}^n)$ of dimension $n^2 = (p+q)^2$. The same is true of $M_0 \subset \mathrm{GL}(U_0) = \mathrm{GL}(\mathbb{C}^{p+q})$.

We now introduce the direct products $P := P_1 \times P_0$ and $M := M_1 \times M_0$, and view these as defining a principal bundle with structure group $G_{\mathbb{R}} = \mathrm{U}_n \times \mathrm{U}_{p,q}$:

$$P \to M \simeq P/G_{\mathbb{R}} \; . \tag{4.10}$$

Since $G_{\mathbb{R}}$ acts on the odd part $\mathfrak{g}_1 \equiv \mathfrak{gl}(U)_1$ of the Lie superalgebra $\mathfrak{gl}(U)$ by Ad, one could associate with $\mathfrak{g}_1$ a complex vector bundle $P \times_{G_{\mathbb{R}}} \mathfrak{g}_1 \to M$, which would give rise to a cs-manifold [**10**] (meaning a supermanifold which is complex in the odd direction). A more concise framework is achieved, however, by passing to a *real* subspace of $\mathfrak{g}_1$.

4.1.2. Real form of $\mathfrak{gl}(U)$. — Write $X \in \mathfrak{gl}(U) = \mathfrak{gl}(U_1 \oplus U_0)$ in block decomposition with respect to the Hermitian vector spaces U_1 and U_0 as

$$X = \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & \mathsf{D} \end{pmatrix} ,$$

where $\mathsf{A} \in \mathrm{End}(U_1)$, $\mathsf{B} \in \mathrm{Hom}(U_0, U_1)$, $\mathsf{C} \in \mathrm{Hom}(U_1, U_0)$, and $\mathsf{D} \in \mathrm{End}(U_0)$. Then define a $\mathbb{Z}_2$-graded real vector space $\mathfrak{u}(U^s)$ as the subset of elements $X \in \mathfrak{gl}(U)$ whose blocks satisfy the set of conditions

$$\begin{aligned} &\langle \mathsf{A} v_1, v_1' \rangle + \langle v_1, \mathsf{A} v_1' \rangle = 0 \,, \quad \langle \mathsf{D} v_0 , s v_0' \rangle + \langle v_0 , s\mathsf{D} v_0' \rangle = 0 \,, \\ &\langle \mathsf{C} v_1, s v_0 \rangle + \mathrm{i} \langle v_1, \mathsf{B} v_0 \rangle = 0 \quad (v_1, v_1' \in U_1;\, v_0 , v_0' \in U_0) \,. \end{aligned} \tag{4.11}$$

This subspace $\mathfrak{u}(U^s) \subset \mathfrak{gl}(U)$ is readily seen to close under the supercommutator, and thus is a Lie sub-superalgebra of $\mathfrak{gl}(U)$. The alternative notation

$$\mathfrak{u}(U^s) \equiv \mathfrak{u}_{n|p,q} \tag{4.12}$$

makes it evident that $\mathfrak{u}(U^s)$ is a pseudo-unitary real form of $\mathfrak{gl}(U) = \mathfrak{gl}_{n|n}$ with mixed signature (p,q) in the non-compact sector $\mathrm{End}(U_0)$.

The even part of the Lie superalgebra $\mathfrak{u}(U^s) = \mathfrak{u}_{n|p,q}$ is the real Lie algebra $\mathfrak{u}_n \oplus \mathfrak{u}_{p,q}$ where $\mathfrak{u}_n = \mathrm{Lie}(\mathrm{U}_n)$ and $\mathfrak{u}_{p,q} = \mathrm{Lie}(\mathrm{U}_{p,q})$. We now focus on the odd component, $\mathfrak{u}(U^s)_1$. From the third condition in (4.11) an element X of the odd space $\mathfrak{u}(U^s)_1$ is already determined by its block $\mathsf{B} \in \mathrm{Hom}(U_0, U_1)$. We therefore have a canonical isomorphism

$$\mathfrak{u}(U^s)_1 \simeq \mathrm{Hom}(U_0, U_1) \simeq U_1 \otimes U_0^* \simeq U_1 \otimes U_0 \,, \tag{4.13}$$

where the last identification is by the pseudo-unitary structure of U_0. This isomorphism $\mathfrak{u}(U^s)_1 \simeq U_1 \otimes U_0$ makes it particularly clear that $\mathfrak{u}(U^s)_1$ is a module for our real Lie group $G_\mathbb{R} = \mathrm{U}(U_1) \times \mathrm{U}(U_0^s) = \mathrm{U}_n \times \mathrm{U}_{p,q}$. In the following we use the simplified notation $\mathfrak{g} := \mathfrak{gl}(U)$ and $\mathfrak{g}_\mathbb{R} := \mathfrak{u}(U^s)$ and

$$\mathfrak{g}_{\mathbb{R},\tau} := \mathfrak{u}(U^s)_\tau \quad (\tau = 0, 1) \,. \tag{4.14}$$

Note that $\mathfrak{g}_{\mathbb{R},0} = \mathrm{Lie}(G_\mathbb{R}) = \mathfrak{u}_n \oplus \mathfrak{u}_{p,q}$.

4.1.3. Lie supergroup structure. — Given the $G_\mathbb{R}$-principal bundle $P \to M$ and the $G_\mathbb{R}$-module $\mathfrak{g}_{\mathbb{R},1}$, we form the associated vector bundle

$$F = P \times_{G_\mathbb{R}} \mathfrak{g}_{\mathbb{R},1} \to M \,. \tag{4.15}$$

Then, following the principles laid down in Sect. 3 we are led to consider $\Gamma(M, \wedge F^*)$, the graded-commutative algebra of sections of the bundle $\wedge(F^*) \to M$.

The key idea now is to extend our function $t \mapsto \chi(t)$ to a radial section of $\Gamma(M, \wedge F^*)$ and exploit the argument of Sect. 3.5, by which the primitive character χ is an eigenfunction of the ring of $\mathfrak{g}$-invariant differential operators. This argument, however, relies on Schur's lemma (which holds over the complex numbers) while the vector bundle $F \to M$ is a real vector bundle over the real manifold M. Thus the good object to consider isn't $\Gamma(M, \wedge F^*)$ but the bundle of *complex* exterior algebras,

$$\mathscr{F} := \Gamma(M, \mathbb{C} \otimes \wedge F^*) \,. \tag{4.16}$$

There, we can implement the basic setting of Sect. 3.5, namely independent left and right actions by the complex Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(U) = \mathfrak{gl}_{n|n}$, as follows.

Recall $M = M_1 \times M_0$ with $M_1 = \mathrm{U}_n$, and let Φ be a section of $\mathscr{F}$. If $X \in \mathfrak{u}_n$ and $\mathrm{e}^{tX} = g \in \mathrm{U}_n$ then from

$$(L_g\Phi)(m_1, m_0) = \Phi(g^{-1}m_1, m_0) \ , \quad (R_g\Phi)(m_1, m_0) = \Phi(m_1 g, m_0) \ ,$$

one gets $\mathfrak{u}_n$-actions on the left and right by taking the differential as usual:

$$\widehat{X}^L := \frac{d}{dt} L_{\exp(tX)}\Big|_{t=0} \ , \quad \widehat{X}^R := \frac{d}{dt} R_{\exp(tX)}\Big|_{t=0} \ .$$

These extend complex linearly to left and right actions of $\mathfrak{gl}(U_1) = \mathfrak{gl}_n = \mathfrak{u}_n + \mathrm{i}\mathfrak{u}_n$:

$$\widehat{X + \mathrm{i}Y}^j := \widehat{X}^j + \mathrm{i}\widehat{Y}^j \quad (j = L, R) \ . \tag{4.17}$$

Next let $h \in \mathrm{H}_{p,q}$. The twisted $\mathrm{H}_{p,q}$-action on M_0 by $m_0 \mapsto h m_0 \sigma(h)^{-1}$ induces an action $h \mapsto T_h$ on sections $\Phi \in \mathscr{F}$ by

$$(T_h\Phi)(m_1, m_0) = \Phi(m_1, h^{-1} m_0 \sigma(h)) \ .$$

To pass to the infinitesimal action, notice that the semigroup $\mathrm{H}_{p,q}$ is open. Therefore, having fixed any point $h \in \mathrm{H}_{p,q}$ one can find $\varepsilon > 0$ so that gh is in $\mathrm{H}_{p,q}$ for every g in the ε-ball $B_\varepsilon \subset \mathrm{GL}(U_0)$ centered at the neutral element of $\mathrm{GL}(U_0)$. This property gets transferred from $\mathrm{H}_{p,q}$ to $M_0 \simeq \mathrm{H}_{p,q}/\mathrm{U}_{p,q}$ by the projection $h \mapsto h\sigma(h)^{-1}$. Thus, for $X \in \mathfrak{gl}(U_0)$ the definition

$$(\widehat{X}\Phi)(m_1, m_0) := \frac{d}{dt} \Phi(m_1, \mathrm{e}^{-tX} m_0 \, \sigma(\mathrm{e}^{tX}))\Big|_{t=0}$$

makes sense, and $X \mapsto \widehat{X}$ is a $\mathfrak{gl}(U_0)$-action on $\mathscr{F}$. Since the assignment $X \mapsto \widehat{X}$ is complex linear on the left and (via σ) complex anti-linear on the right, one gets from it a left action $X \mapsto \widehat{X}^L$ and a right action $X \mapsto \widehat{X}^R$ by setting

$$\widehat{X}^L := \tfrac{1}{2}(\widehat{X} - \mathrm{i}\,\widehat{(\mathrm{i}X)}) \ , \quad \widehat{X}^R := \tfrac{1}{2}(\widehat{X} + \mathrm{i}\,\widehat{(\mathrm{i}X)}) \ . \tag{4.18}$$

These two actions commute, i.e., $[\widehat{X}^L, \widehat{Y}^R] = 0$ for all $X, Y \in \mathfrak{gl}(U_0)$. Altogether we now have left and right actions of the even part $\mathfrak{g}_0 \simeq \mathfrak{gl}(U_1) \oplus \mathfrak{gl}(U_0)$ of our complex Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(U)$.

To describe the action of the odd part, $\mathfrak{g}_1 = \mathfrak{gl}(U)_1$, we first of all adapt our notation. Although the $G_\mathbb{R}$-principal bundle P was introduced as

$$P = P_1 \times P_0 = (\mathrm{U}_n \times \mathrm{U}_n) \times (\mathrm{H}_{p,q} \overset{1,\sigma}{\hookrightarrow} \mathrm{GL}(U_0) \times \mathrm{GL}(U_0)) \ ,$$

we now rearrange factors to interpret P as a subset of the direct product of two copies of $G = (\mathrm{GL}(U_1) \times \mathrm{GL}(U_0) \overset{\mathrm{diag}}{\hookrightarrow} \mathrm{GL}(U))$:

$$P \subset G \times G \ ,$$

and write $p = (x, y)$ for the points of P accordingly. Such an element $(x, y) \in P$ consists of $x = \mathrm{diag}(u_L, h)$ and $y = \mathrm{diag}(u_R, \sigma(h))$ where $h \in \mathrm{H}_{p,q}$ and $u_L, u_R \in \mathrm{U}_n$.

Recalling from Sect. 3.4 the meaning of the tautological object

$$\xi = \sum \varphi^i \otimes \mathsf{F}_i \in \mathfrak{g}_{\mathbb{R},1}^* \otimes \mathfrak{g}_{\mathbb{R},1} ,$$

where $\{\mathsf{F}_i\}$ is some basis of $\mathfrak{g}_{\mathbb{R},1} = \mathfrak{u}(U^s)_1$, and the dual basis $\{\varphi^i\}$ is viewed as a set of generators of $\wedge(\mathfrak{g}_{\mathbb{R},1}^*)$, we then consider the supermatrix

$$P \ni (x,y) \mapsto \Xi = x\,\mathrm{e}^{\xi} y^{-1} . \tag{4.19}$$

From the relation $x\,\mathrm{e}^{\xi} y^{-1} = (xg)\,\mathrm{e}^{\sum \varphi^i \mathrm{Ad}^*(g^{-1})_i^{\;j} \otimes \mathsf{F}_j} (yg)^{-1}$ for $g \in G_\mathbb{R}$, the matrix entries of Ξ are $G_\mathbb{R}$-equivariant functions $P \to \wedge(\mathfrak{g}_{\mathbb{R},1}^*)$ and, thus, sections of $\mathscr{F}$.

The rest of the development exactly follows Sect. 3.2.3 and we indicate it only very briefly. Enlarging the complex Grassmann algebra $\wedge(\mathfrak{g}_1^*) = \mathbb{C} \otimes \wedge(\mathfrak{g}_{\mathbb{R},1}^*)$ by an anti-commuting parameter η one shows that (see Lem. 3.1 and its proof) if Y is an odd generator $Y \in \mathfrak{g}_1 = \mathfrak{gl}(U)_1$ there exist even-type functions $y \mapsto \alpha_Y^i(y) \in \wedge^{\mathrm{even}}(\mathfrak{g}_1^*)$ and odd-type functions $y \mapsto \beta_Y^j(y) \in \wedge^{\mathrm{odd}}(\mathfrak{g}_1^*)$ so that

$$(x\,\mathrm{e}^{\xi} y^{-1})\mathrm{e}^{\eta \otimes Y} = x\,\mathrm{e}^{\sum(\varphi^i + \alpha_Y^i(y)\eta) \otimes \mathsf{F}_i} y^{-1} \mathrm{e}^{\sum \beta_Y^j(y)\eta \otimes \mathsf{E}_j} ,$$

where $\{\mathsf{E}_j\}$ is a basis of $\mathfrak{g}_{\mathbb{R},0}$. By the general principle explained in Prop. 3.2 and its proof, these functions determine a right action $Y \mapsto \widehat{Y}^R$ of $\mathfrak{g}_1 = \mathfrak{gl}(U)_1$. The left action $Y \mapsto \widehat{Y}^L$ is constructed in the same fashion.

Let us now summarize the material of this subsection.

Proposition 4.2. — *The complex Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(U)$ acts on the graded-commutative algebra $\mathscr{F} = \Gamma(M, \mathbb{C} \otimes \wedge F^*)$ on the left and right by $X \mapsto \widehat{X}^L$ resp. $X \mapsto \widehat{X}^R$.*

4.2. Extending the character χ. — Note that if Ξ is decomposed as $\Xi = \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & \mathsf{D} \end{pmatrix}$ according to $U = U_1 \oplus U_0$, the matrix entries of the blocks A and D are even sections of $\mathscr{F}$, and those of the blocks B and C are odd sections.

For the following definition recall $V = U \otimes \mathbb{C}^N$, and let $u \in \mathrm{U}_N$.

Definition 4.3. — *By the reciprocal of the superdeterminant of* $\mathrm{Id}_V - \Xi \otimes u$ *we mean*

$$\mathrm{SDet}(\mathrm{Id}_V - \Xi \otimes u)^{-1} = \frac{\mathrm{Det}\big(\mathrm{Id}_{V_1} - \mathsf{A} \otimes u + (\mathsf{B} \otimes u)\,(\mathrm{Id}_{V_0} - \mathsf{D} \otimes u)^{-1}(\mathsf{C} \otimes u)\big)}{\mathrm{Det}(\mathrm{Id}_{V_0} - \mathsf{D} \otimes u)} .$$

Remark. — This is Def. 1.5 (with Prop. 1.6) transcribed to the present setting. We will show later, by Lem. 4.8, that $\mathrm{Id}_{V_0} - \mathsf{D} \otimes u$ has an inverse for all $u \in \mathrm{U}_N$ and $\mathsf{D} = \Xi|_{U_0 \to U_0}$, thereby ensuring the *global* existence of $\mathrm{SDet}(\mathrm{Id}_V - \Xi \otimes u)^{-1}$.

Our goal is to relate the character $t \mapsto \chi(t)$ to a section of $\mathscr{F}$. To that end, let us reorganize the notation of Cor. 2.4 so that $t = (t_1, t_0) \in \mathrm{GL}(U_1) \times \mathrm{GL}(U_0)$ with

$$t_1 = t_1^+ + t_1^- = \sum_{k=1}^{p+q} \mathrm{e}^{\mathrm{i}\psi_k} E_{kk}^{11} , \quad t_0 = t_0^+ + t_0^- = \sum_{k=1}^{p+q} \mathrm{e}^{\phi_k} E_{kk}^{00} , \tag{4.20}$$

where $\{E_{ij}^{\tau\tau'}\}$ is a standard basis of $\mathrm{End}(U_1 \oplus U_0)$. Although the parameters ψ_k and ϕ_k previously assumed values in $\mathbb{C}$, we now *restrict their range to the real numbers*, while retaining (from Cor. 2.4) the condition

$$\phi_j < 0 < \phi_l \quad (1 \le j \le p < l \le p+q) . \tag{4.21}$$

Imposing these restrictions, we may regard the diagonal transformations $t = (t_1, t_0)$ as points of $M = M_1 \times M_0$. Indeed, t_1 now lies in $\mathrm{U}_1^n \subset \mathrm{U}_n = M_1$; and it is easy to see that $t_0 \in \mathrm{H}_{p,q}$ and $t_0 = \sigma(t_0)^{-1}$, and therefore we have $t_0 \in M_0$.

Thus we now choose to view our character $t \mapsto \chi(t)$ as a function of the diagonal transformations $t \in M$. At the same time, we restrict the assignment of (2.24), i.e.,

$$t \mapsto \rho(t) = R(t \otimes \mathrm{Id}_N) ,$$

to the domain for t as above. Then from Prop. 2.10 we still have the expression

$$\chi(t) = \mathrm{STr}_{\mathscr{V}_\lambda}\, \rho(t) = \int_{\mathrm{U}_N} \frac{\mathrm{Det}(\mathrm{Id} - t_1 \otimes u)}{\mathrm{Det}(\mathrm{Id} - t_0 \otimes u)}\, du . \tag{4.22}$$

As a preparatory step for Prop. 4.4 below, recall from Sect. 2.6 the representation

$$\rho_* :\ \mathfrak{gl}_{n|n} \to \mathfrak{gl}(\mathscr{V}_\lambda) ,$$

and from Sect. 3.4 that by the character of a representation $(\mathscr{V}_\lambda, \rho_*, \rho)$ of a Lie supergroup $\mathfrak{G} = (\mathfrak{g}, G)$ with supermatrix $\Xi = x\,\mathrm{e}^{\xi} y^{-1} = x\,\mathrm{e}^{\sum \varphi^i \otimes \mathsf{F}_i} y^{-1}$, we mean

$$\mathrm{STr}_{\mathscr{V}_\lambda}\, \rho(x)\, \mathrm{e}^{\sum \varphi^i \rho_*(\mathsf{F}_i)} \rho(y^{-1}) \equiv \mathrm{STr}_{\mathscr{V}_\lambda}\, \rho(\Xi) . \tag{4.23}$$

In the present refined setting, where the G-principal bundle $G \times G \to G$, $(g,h) \mapsto gh^{-1}$ has been replaced by the $G_\mathbb{R}$-principal bundle

$$G \times G \supset P \to M \subset G , \quad (x,y) \mapsto xy^{-1} ,$$

and the associated complex vector bundle $(G \times G) \times_G \mathfrak{g}_1 \to G$ by

$$F = P \times_{G_\mathbb{R}} \mathfrak{g}_{\mathbb{R},1} \to M ,$$

we still mean the same thing.

Proposition 4.4. — *The assignment $t \mapsto \rho(t)$ extends to a semigroup representation*

$$\rho :\ \mathrm{GL}_n \times \mathrm{H}_{p,q} \ \to\ \mathrm{GL}(\mathscr{V}_\lambda) ,$$

and a corresponding Lie group representation

$$\rho' :\ G_\mathbb{R} = \mathrm{U}_n \times \mathrm{U}_{p,q} \ \to\ \mathrm{U}(\mathscr{V}_\lambda) ,$$

which are compatible with the $\mathfrak{gl}_{n|n}$-representation ρ_*. The character associated with $(\mathscr{V}_\lambda, \rho_*, \rho)$ by equation (4.23) exists globally as an analytic section of the algebra $\mathscr{F} = \Gamma(M, \mathbb{C} \otimes \wedge F^*)$, and is given by an extension of the integral formula (4.22):

$$\mathrm{STr}_{\mathscr{V}_\lambda}\, \rho(\Xi) = \int_{\mathrm{U}_N} \mathrm{SDet}(\mathrm{Id} - \Xi \otimes u)^{-1} du . \tag{4.24}$$

Remark. — ρ' corresponds to ρ in the sense that $\rho(xg) = \rho(x)\rho'(g)$ and $\rho((yg)^{-1}) = \rho'(g)^{-1}\rho(y^{-1})$ for $g \in G_\mathbb{R}$ and $x, y^{-1} \in \mathrm{GL}_n \times \mathrm{H}_{p,q}$. These relations and the compatibility of representations guarantee that $(x,y) \mapsto \mathrm{STr}_{\mathscr{V}_\lambda}\, \rho(x)\, \mathrm{e}^{\sum \varphi^i \rho_*(\mathsf{F}_i)} \rho(y^{-1})$ is a $G_\mathbb{R}$-equivariant mapping and hence defines a section $\chi \in \mathscr{F}$ in the usual way:

$$\chi(xy^{-1}) = \left[(x,y) \, ; \, \mathrm{STr}_{\mathscr{V}_\lambda}\, \rho(x)\, \mathrm{e}^{\sum \varphi^i \rho_*(\mathsf{F}_i)} \rho(y^{-1})\right] .$$

Although it might seem that Prop. 4.4 extends the character formula of Cor. 4.1 in a straightforward manner, we do not know any easy way of proving the existence of the representations ρ and ρ'; and to avoid a lengthy interruption in the flow of our argument, we must postpone the proof of Prop. 4.4 until the end of Sect. 5.

4.3. All Casimir invariants vanish on $\mathscr{V}_\lambda$. — Prop. 4.4 achieves the important step of extending the torus function $t \mapsto \chi(t)$ to a section of the graded-commutative algebra $\mathscr{F} = \Gamma(M, \mathbb{C} \otimes \wedge F^*)$. Hence, taking Prop. 4.4 for granted, we can apply the powerful machinery reviewed in Sect. 3: the Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(U) = \mathfrak{gl}_{n|n}$ acts on $\mathscr{F}$ by left- or right-invariant derivations; this induces two actions of the universal enveloping algebra $\mathsf{U}(\mathfrak{gl}_{n|n})$ on $\mathscr{F}$; and the Casimir invariants, i.e. the central elements of $\mathsf{U}(\mathfrak{gl}_{n|n})$, then act on $\mathscr{F}$ by $\mathfrak{gl}_{n|n}$-invariant differential operators, the Laplace-Casimir operators. What's most important is that, since the $\mathfrak{gl}_{n|n}$-representation $(\mathscr{V}_\lambda, \rho_*)$ is irreducible, the character χ is an eigenfunction of all of these operators.

The next step is to calculate the Laplace-Casimir eigenvalues. To that end, recall from Lem. 3.6 the expression for the degree-ℓ Casimir invariant I_ℓ of $\mathfrak{gl}_{n|n}$. There exists a simple heuristic why all of these invariants must be identically zero in such a representation as $(\mathscr{V}_\lambda, \rho)$. (In this subsection we use the simplified notation $\rho_* \equiv \rho$.)

First of all, one argues that each invariant I_ℓ can be expressed as an anti-commutator of two odd elements of the universal enveloping algebra $\mathsf{U}(\mathfrak{gl}_{n|n})$. Indeed, using the basic bracket relations (3.1) of $\mathfrak{gl}_{n|n}$ it is straightforward to verify that

$$I_\ell = [Q, F^{(\ell)}] ,$$

where Q, $F^{(\ell)} \in \mathsf{U}(\mathfrak{gl}_{n|n})$ are given in terms of a standard basis $\{E_{ij}^{\tau\tau'}\}$ by

$$Q = \sum_{i=1}^{n} E_{ii}^{10} , \quad F^{(\ell)} = \sum E_{ij_2}^{0\,\tau_2} (-1)^{\tau_2} E_{j_2 j_3}^{\tau_2 \tau_3} (-1)^{\tau_3} \cdots E_{j_{\ell-1} j_\ell}^{\tau_{\ell-1} \tau_\ell} (-1)^{\tau_\ell} E_{j_\ell i}^{\tau_\ell 1} .$$

If the representation space $\mathscr{V}_\lambda$ were finite-dimensional, we could now argue that

$$\mathrm{STr}_{\mathscr{V}_\lambda}\, \rho(I_\ell) = \mathrm{STr}_{\mathscr{V}_\lambda}\, [\rho(Q), \rho(F^{(\ell)})] = 0 ,$$

since the supertrace of any bracket vanishes. On the other hand, since I_ℓ is a Casimir invariant, the operator $\rho(I_\ell)$ on the irreducible representation space $\mathscr{V}_\lambda$ must be a multiple of unity: $\rho(I_\ell) = \alpha(I_\ell) \times \mathrm{Id}_{\mathscr{V}_\lambda}$. In finite dimension we could therefore say that

$$\mathrm{STr}_{\mathscr{V}_\lambda}\, \rho(I_\ell) = \alpha(I_\ell)\, \mathrm{STr}_{\mathscr{V}_\lambda}\, \mathrm{Id} = \alpha(I_\ell)(\dim \mathscr{V}_{\lambda,0} - \dim \mathscr{V}_{\lambda,1}) .$$

Inspection of $\mathscr{V}_\lambda$ shows that there is exactly one vector in its even subspace which has no partner in the odd subspace – this vector is the "vacuum". In this sense, the dimension of $\mathscr{V}_{\lambda,0}$ exceeds that of $\mathscr{V}_{\lambda,1}$ by one. Hence $\mathrm{STr}_{\mathscr{V}_\lambda}\, \rho(I_\ell) = 1 \cdot \alpha(I_\ell)$, and

from the previous result $\mathrm{STr}_{\mathscr{V}_\lambda}\,\rho(I_\ell) = 0$ we would be forced to conclude $\alpha(I_\ell) = 0$. Let us record this conclusion for later use.

Lemma 4.5. — *In an irreducible $\mathfrak{gl}_{n|n}$-representation on a finite-dimensional vector space V with $\dim V_0 \neq \dim V_1$, all Casimir invariants I_ℓ of $\mathfrak{gl}_{n|n}$ are identically zero.*

While the argument leading to Lem. 4.5 is correct for finite dimension, it is not sound for the case of our infinite-dimensional representation space $\mathscr{V}_\lambda$, as the traces $\mathrm{STr}_{\mathscr{V}_\lambda}\,\rho(I_\ell)$ and $\mathrm{STr}_{\mathscr{V}_\lambda}\,\mathrm{Id}$ do not converge. Nevertheless, the conclusion still holds true:

Proposition 4.6. — *In the representation $(\mathscr{V}_\lambda, \rho)$ all Casimir invariants I_ℓ vanish.*

Proof. — The heuristic argument becomes rigorous on regularizing the traces. This is done with the help of a difference of two sums of generators,

$$\Lambda = \sum_{\tau=0}^{1}\sum_{j=1}^{p} E_{jj}^{\tau\tau} - \sum_{\tau=0}^{1}\sum_{l=1}^{q} E_{l+p,l+p}^{\tau\tau} ,$$

which is not a Casimir invariant of $\mathfrak{gl}_{n|n}$, but does lie in the center of $\mathsf{U}(\mathfrak{gl}_{p|p} \oplus \mathfrak{gl}_{q|q})$. It commutes with both Q and $F^{(\ell)}$ because these respect the vector space decomposition $\mathbb{C}^n = \mathbb{C}^p \oplus \mathbb{C}^q$. On $\mathscr{V}_\lambda$ it is represented by the operator

$$\begin{aligned} \rho(\Lambda) \;&=\; \sum_{j=1}^{p}\sum_{a=1}^{N} \big(\varepsilon(e_{1,j}^{+} \otimes e_a)\iota(f_{+}^{1,j} \otimes f^a) + \mu(e_{0,j}^{+} \otimes e_a)\delta(f_{+}^{0,j} \otimes f^a)\big) \\ &+\; \sum_{l=1}^{q}\sum_{a=1}^{N} \big(\varepsilon(f_{-}^{1,l} \otimes f^a)\iota(e_{1,l}^{-} \otimes e_a) + \mu(f_{-}^{0,l} \otimes f^a)\delta(e_{0,l}^{-} \otimes e_a)\big) , \end{aligned}$$

where the notation of Sect. 2 is being used. Λ is called the particle number in physics.

Now if t is any positive real parameter, inserting the operator $\mathrm{e}^{-t\,\rho(\Lambda)}$ cuts off the infinite contribution from high particle numbers and makes the traces of the heuristic argument converge. Thus, on the one hand we now rigorously have

$$\mathrm{STr}_{\mathscr{V}_\lambda}\,\mathrm{e}^{-t\,\rho(\Lambda)}\rho(I_\ell) = \mathrm{STr}_{\mathscr{V}_\lambda}\,[\rho(Q), \mathrm{e}^{-t\,\rho(\Lambda)}\rho(F^{(\ell)})] = 0 ,$$

and on the other hand, since $\rho(I_\ell) = \alpha(I_\ell) \times \mathrm{Id}_{\mathscr{V}_\lambda}$,

$$\mathrm{STr}_{\mathscr{V}_\lambda}\,\mathrm{e}^{-t\,\rho(\Lambda)}\rho(I_\ell) = \alpha(I_\ell)\,\mathrm{STr}_{\mathscr{V}_\lambda}\,\mathrm{e}^{-t\,\rho(\Lambda)} .$$

To compute the last trace, notice that $\mathrm{e}^{-t\Lambda}$ is the diagonal transformation

$$\mathrm{e}^{-t\Lambda} = \mathrm{e}^{-t}\sum E_{jj}^{\tau\tau} + \mathrm{e}^{t}\sum E_{l+p,l+p}^{\tau\tau} ,$$

and use the formula (4.22) to obtain

$$\mathrm{STr}_{\mathscr{V}_\lambda}\,\mathrm{e}^{-t\,\rho(\Lambda)} = \int_{\mathrm{U}_N} \frac{\mathrm{Det}^p(\mathrm{Id} - \mathrm{e}^{-t}u)\,\mathrm{Det}^q(\mathrm{Id} - \mathrm{e}^{t}u)}{\mathrm{Det}^p(\mathrm{Id} - \mathrm{e}^{-t}u)\,\mathrm{Det}^q(\mathrm{Id} - \mathrm{e}^{t}u)}\,du = 1 .$$

Therefore

$$0 = \mathrm{STr}_{\mathscr{V}_\lambda}\,\mathrm{e}^{-t\,\rho(\Lambda)}\rho(I_\ell) = \alpha(I_\ell) ,$$

and the proposition is proved. □

Remark. — Up to this point, everything would have gone through for the most general case of ratios (1.1) with an arbitrary number of characteristic polynomials and their complex conjugates in the numerator and denominator. However, we now had to require these numbers to be pairwise equal, or else Prop. 4.6 would have been false.

As an immediate consequence of Prop. 4.6 and the relationship between Casimir invariants I_ℓ and Laplace-Casimir operators $D(I_\ell)$, we have:

Corollary 4.7. — *The irreducible* $\mathrm{GL}_{n|n}$*-character* $\chi(\Xi) = \mathrm{STr}_{\mathscr{V}_\lambda}\, \rho(\Xi)$ *lies in the kernel of the ring of* $\mathfrak{gl}_{n|n}$*-invariant differential operators* $D(I_\ell)$:

$$D(I_\ell)\chi = 0 \quad (\ell \in \mathbb{N}) \; .$$

Remark. — Thus the $\mathfrak{gl}_{n|n}$-representation $(\mathscr{V}_\lambda, \rho)$ is *degenerate* with the trivial representation (the Laplace-Casimir eigenvalues are the same), and is *atypical*. Sadly, the characters of atypical representations are not covered by the known generalization [**2**] of the Weyl character formula to the case of Lie supergroups. To compute the character $\chi(t) = \mathrm{STr}_{\mathscr{V}_\lambda}\, \rho(t)$, further effort is required.

4.4. Maximal torus. — In Sect. 3.8 we reviewed some of Berezin's results for the radial parts of the Laplace-Casimir operators $D(I_\ell)$ for $\mathfrak{gl}_{m|n}$. To bring these to bear on our problem, we first need to establish the existence of a kind of maximal torus for the real-analytic manifold $M = M_1 \times M_0$. The main step here is the following lemma.

Its proof needs two basic facts (established by Cor. 5.3): the semigroup $\mathrm{H}_{p,q}$ is connected, and the elements $h \in \mathrm{H}_{p,q}$ satisfy the inequality $hsh^\dagger < s$ as well as $h^\dagger sh < s$.

Lemma 4.8. — *Each element of* $M_0 = \{h\sigma(h)^{-1} \mid h \in \mathrm{H}_{p,q}\}$ *is brought to diagonal form by a pseudo-unitary transformation* $g \in \mathrm{U}_{p,q}$. *The eigenvalues lie in* $\mathbb{R}_+ \setminus \{1\}$.

Proof. — On general grounds, every $m \in M_0$ has at least one eigenvector $v \neq 0$. Notice that the eigenvalue λ must be non-zero because $m = h\,\sigma(h)^{-1} = hsh^\dagger s$ has an inverse. Now the inequality $hsh^\dagger < s$ for $h \in \mathrm{H}_{p,q}$ leads to

$$\lambda \langle sv, v\rangle = \langle sv, mv\rangle = \langle sv, (hsh^\dagger)sv\rangle < \langle sv, v\rangle \; .$$

One then infers that $\langle v, sv\rangle = \langle sv, v\rangle \neq 0$, and that λ is real and $\lambda \neq 1$. If $\langle v, sv\rangle < 0$ then $\lambda > 1$, and if $\langle v, sv\rangle > 0$ then $\lambda < 1$. In the former case we fix the normalization of v by the condition $\langle v, sv\rangle = -1$ and in the latter case by $\langle v, sv\rangle = 1$.

The pseudo-Hermitian form $\langle\, ,\rangle_s : U_0 \times U_0 \to \mathbb{C}$, $(u, v) \mapsto \langle u, sv\rangle$ is non-degenerate. Therefore, since $\langle v, v\rangle_s \equiv \langle v, sv\rangle \neq 0$, the vector space U_0 decomposes as a direct sum

$$U_0 = \mathbb{C}v \oplus U^\perp$$

where $U^\perp$ is the subspace of U_0 which is $\langle\, ,\rangle_s$-orthogonal to the complex line $\mathbb{C}v$.

Now $m = hsh^\dagger s$ is self-adjoint w.r.t. the pseudo-Hermitian form $\langle\, ,\rangle_s$ and hence leaves the decomposition $U_0 = \mathbb{C}v \oplus U^\perp$ the same. Thus one can split off the eigenspace $\mathbb{C}v$ from U_0 and repeat the whole discussion for m restricted to $U^\perp$. The restriction $m' := m|_{U^\perp}$ has again at least one eigenvector, say v', and $\langle v', v'\rangle_s \neq 0$ by the same argument as before. The eigenvalue λ' is real with $\lambda' \neq 0$ and $\lambda' \neq 1$. Since m' is $\langle\, ,\rangle_s$

self-adjoint and preserves the decomposition of $U^\perp$ by the complex line $\mathbb{C}v'$ plus its orthogonal complement, one can split off $\mathbb{C}v'$ from $U^\perp$. By continuing in this way, one concludes that m is diagonalizable, and that the $(p+q)$-dimensional vector space U_0 decomposes as a $\langle\, ,\rangle_s$-orthogonal direct sum of m-eigenspaces $\mathbb{C}v_1, \ldots, \mathbb{C}v_{p+q}$.

It remains to show that the diagonalizing transformation g lies in the pseudo-unitary group $\mathrm{U}_{p,q}$. For this, let $\{e_i\}_{i=1,\ldots,p+q}$ be an orthonormal basis of U_0 such that $e_1, \ldots, e_p$ span the s-positive subspace U_0^+, and $e_{p+1}, \ldots, e_{p+q}$ span U_0^-. It is clear that if $\lambda^\pm$ are two real numbers in the range $0 < \lambda^+ < 1$ and $1 < \lambda^- < \infty$, then

$$m = \lambda^+ \, \mathrm{Id}_{U_0^+} \oplus \lambda^- \, \mathrm{Id}_{U_0^-}$$

lies in M_0. Indeed, m equals $h\sigma(h)^{-1}$ for $h = \sqrt{\lambda^+}\,\mathrm{Id}_{U_0^+} \oplus \sqrt{\lambda^-}\,\mathrm{Id}_{U_0^-} \in \mathrm{H}_{p,q}$. Note that the signature of $\mathrm{Id}_{U_0} - m$ for this m is the same as the signature of s.

Now, since the manifold $\mathrm{H}_{p,q}$ is connected, so is $M_0 = \{h\sigma(h)^{-1} \mid h \in \mathrm{H}_{p,q}\}$. When m is pushed around in M_0, the eigenvalues of m move but, as we have seen, they are always real and never hit zero or one. Therefore, they stay in $\mathbb{R}_+ \setminus \{1\}$ and the signature of $\mathrm{Id}_{U_0} - m$ for $m \in M_0$ is an invariant and is given by s. By reordering the indices, one can arrange for the positive signature eigenspace of $\mathrm{Id}_{U_0} - m$ to be spanned by the first p eigenvectors $v_1, \ldots, v_p$ of m. A linear transformation $g \in \mathrm{GL}(U_0)$ diagonalizes m if $v_i = ge_i$ for $i = 1, \ldots, p+q$. Since the m-eigenvectors v_i have been constructed to form an orthonormal system with respect to the pseudo-Hermitian form $\langle\, ,\rangle_s$, it follows that g is pseudo-unitary: $g \in \mathrm{U}_{p,q}$. ☐

Remark. — The numerical part of the operator $\mathrm{Id}_{V_0} - \mathsf{D} \otimes u$ of Def. 4.3 is $\mathrm{Id}_{V_0} - m \otimes u$ for $u \in \mathrm{U}_N$ and some $m \in M_0$. Since the eigenvalues of m lie in $\mathbb{R}_+ \setminus \{1\}$, we see that $\mathrm{Id}_{V_0} - \mathsf{D} \otimes u$ in fact has an inverse.

Every element $m_1 \in M_1 = \mathrm{U}_n$ is diagonalizable by a unitary transformation and has unitary eigenvalues. In combination with Lem. 4.8 this has the following consequence for $m = (m_1, m_0) \in M_1 \times M_0 = M$.

Corollary 4.9. — *Let $T = T_1 \times T_0$ with*

$$T_1 = \mathrm{U}_1^n\,, \quad T_0 = (0,1)^p \times (1,\infty)^q \quad (p+q=n)\,,$$

be the set of diagonal transformations $t = (t_1, t_0)$ defined in Eqs. (4.20) and (4.21) with real-valued ψ_k and ϕ_k restricted by $\phi_j < 0 < \phi_l$. This Abelian semigroup T is a maximal torus for $M = M_1 \times M_0$ in the sense that for every $m \in M$ there exist $t \in T$ and $g \in G_\mathbb{R} = \mathrm{U}_n \times \mathrm{U}_{p,q}$ such that

$$g^{-1} m g = t\,.$$

Recall now from Sect. 2.7 that we view our variable parameters ψ_k and ϕ_k as complex linear functions on a complex Cartan subalgebra $\mathfrak{h} \subset \mathfrak{gl}_{n|n}$. Let $\mathfrak{h}_\mathbb{R} \subset \mathfrak{h}$ be the real form defined by the requirement that if $H \in \mathfrak{h}_\mathbb{R}$ then $\psi_k(H) \in \mathbb{R}$ and $\phi_k(H) \in \mathbb{R}$ for $k = 1, \ldots, n$. The condition imposed after Eq. (4.20) can then be rephrased as the statement that our parameters restrict to linear functions

$$\psi_k : \mathfrak{h}_\mathbb{R} \to \mathbb{R} \quad \text{and} \quad \phi_k : \mathfrak{h}_\mathbb{R} \to \mathbb{R} \quad (k = 1, \ldots, n)\,. \tag{4.25}$$

The further conditions $\phi_j(H) < 0 < \phi_l(H)$ for $1 \le j \le p < l \le n$ select an open subspace $\mathfrak{t} \subset \mathfrak{h}_{\mathbb{R}}$ – note that $\mathfrak{t}$ is not a vector space – with the property

$$\exp \mathfrak{t} = T \ . \tag{4.26}$$

4.5. ***W*-invariance. —** From Prop. 4.4 the character $\chi(\Xi)$ exists as an analytic section of $\mathscr{F}$, and this section is radial. Since the domain $M = M_1 \times M_0$ is invariant under conjugation by $g \in G_{\mathbb{R}} = \mathrm{U}_n \times \mathrm{U}_{p,q}$, so is the character. Therefore, if $g \in G_{\mathbb{R}}$ is any element which normalizes T, i.e. $\forall t \in T : gtg^{-1} \in T$, the restriction $\chi : T \to \mathbb{C}$ inherits the invariance property

$$\chi(t) = \chi(gtg^{-1}) \ . \tag{4.27}$$

The transformations $t \mapsto gtg^{-1} =: w \cdot t$ of T arising in this way form the Weyl group W. Since the neutral element $t = e$ is contained in the closure of T, one can differentiate the W-action $t \mapsto w \cdot t$ at the fixed point $t = e$ to obtain an induced linear action $H \mapsto \mathrm{Ad}(g)H =: w(H)$ of W on tangent vectors $H \in \mathfrak{h}_{\mathbb{R}}$. This action in turn induces an action of W on the linear functions $f : \mathfrak{h}_{\mathbb{R}} \to \mathbb{C}$ by $w(f) = f \circ w^{-1}$.

Proposition 4.10. — *The Weyl group of our problem is $W = \mathrm{S}_n \times (\mathrm{S}_p \times \mathrm{S}_q)$ where the symmetric group S_n permutes the $n = p + q$ functions*

$$(\psi_1, \ldots, \psi_n) \ ,$$

while the first and second factor in $\mathrm{S}_p \times \mathrm{S}_q$ permute the first p resp. last q entries of

$$(\phi_1, \ldots, \phi_p, \phi_{p+1}, \ldots, \phi_{p+q}) \ .$$

The character $\chi(t) = \mathrm{STr}_{\mathscr{V}_\lambda}\, \rho(t)$ satisfies $\chi(t) = \chi(w \cdot t)$ for all $w \in W$.

Proof. — Let $H = (H_1, H_0) \in \mathfrak{t} \subset \mathfrak{h}_{\mathbb{R}}$ be a regular point, i.e., H viewed as a diagonal linear transformation of $\mathbb{C}^{n|n}$ has eigenvalues $\mathrm{i}\psi_1(H), \ldots, \phi_n(H)$ which are pairwise distinct and non-zero. Consider the orbit of such a point H under the adjoint action of $G_{\mathbb{R}} = \mathrm{U}_n \times \mathrm{U}_{p,q}$. Every distinct intersection of this orbit with $\mathfrak{t}$ amounts to one element of the Weyl group $W = W_1 \times W_0$. Since conjugation by $g \in G_{\mathbb{R}}$ leaves the eigenvalues of H unchanged, every point of intersection must correspond to a permutation of these eigenvalues. The n eigenvalues $\mathrm{i}\psi_1(H), \ldots, \mathrm{i}\psi_n(H)$ are imaginary and conjugation by $(u_1, e) \in G_{\mathbb{R}}$ with suitably chosen $u_1 \in \mathrm{U}_n$ allows to arbitrarily permute them; hence $W_1 = \mathrm{S}_n$. The $n = p + q$ eigenvalues $\phi_1(H), \ldots, \phi_n(H)$ are real, with the first p being negative ($\phi_j < 0$) and the last q positive ($\phi_l > 0$). Conjugation by (e, u_0) with $u_0 \in \mathrm{U}_{p,q}$ never mixes these two sets but only permutes them separately. Thus $W_0 = \mathrm{S}_p \times \mathrm{S}_q$.

The W-invariance $\chi(t) = \chi(w \cdot t)$ is a restatement of Eq. (4.27). ☐

4.6. Radial differential equations for χ. — Being the character (or supertrace) of a representation, the analytic section $\chi \in \mathscr{F}$ is radial. Hence, if $\dot{D}(I_\ell)$ is the radial part of the $\mathfrak{gl}_{n|n}$-invariant differential operator $D(I_\ell)$, the equation $D(I_\ell)\chi = 0$ reduces to $\dot{D}(I_\ell)\chi(t) = 0$ for the W-invariant torus function $t \mapsto \chi(t)$.

We now write the system ($\ell \in \mathbb{N}$) of differential equations $\dot{D}(I_\ell)\chi(t) = 0$ in explicit form by drawing on Berezin's results (Thm. 3.9 of Sect. 3). For that purpose we fix a

set $\Delta^+ = \Delta_0^+ \cup \Delta_1^+$ of positive roots of the Lie superalgebra $\mathfrak{gl}_{n|n}$. By the same choice that was made in Sect. 2.7 before, we take the even resp. odd positive roots to be

$$\begin{aligned} \Delta_0^+ : &\quad \mathrm{i}\psi_k - \mathrm{i}\psi_{k'} \,, \quad \phi_k - \phi_{k'} \quad (1 \le k' < k \le n)\,, \\ \Delta_1^+ : &\quad \mathrm{i}\psi_k - \phi_j \,, \quad \phi_l - \mathrm{i}\psi_k \quad (1 \le j \le p < l \le n)\,. \end{aligned}$$

Note that if $H \in \mathfrak{t}$, then $\mathfrak{Re}\,\beta(H) > 0$ for all roots $\beta \in \Delta_1^+$.

While we have chosen to regard our parameters ψ_k and ϕ_k as linear functions on $\mathfrak{h}_\mathbb{R} \supset \mathfrak{t}$, another possibility would have been to use composition with the logarithm to define them as *local coordinates* on $T = \exp \mathfrak{t}$. From the latter perspective, they determine differential operators $\partial/\partial\psi_k$ and $\partial/\partial\phi_k$ on functions $f : T \to \mathbb{C}$.

Let D_ℓ as in Sect. 3.8 (after replacing $\phi_k \to -\mathrm{i}\phi_k$) be the differential operator

$$D_\ell = \sum_{k=1}^n \frac{\partial^\ell}{\partial\psi_k^\ell} - (-\mathrm{i})^\ell \sum_{k=1}^n \frac{\partial^\ell}{\partial\phi_k^\ell}\,, \tag{4.28}$$

and note that D_ℓ is W-invariant, i.e., commutes with the W-action on functions $f(t)$. Notice also that the degree-2 operator D_2 is elliptic.

Proposition 4.11. — *The character $\chi : T \to \mathbb{C}$ given by $\chi(t) = \mathrm{STr}_{\mathscr{V}_\lambda}\,\rho(t)$ satisfies the set of differential equations (defined on the set of regular points $T' \subset T$)*

$$J^{-1/2} D_\ell \big(J^{1/2}\chi\big) = 0 \quad (\ell \in \mathbb{N})\,,$$

where the function $J^{1/2} : T' \to \mathbb{C}$ is a square root of

$$J(t) = \frac{\prod_{\alpha\in\Delta_0^+} \sinh^2\left(\frac{1}{2}\alpha(\ln t)\right)}{\prod_{\beta\in\Delta_1^+} \sinh^2\left(\frac{1}{2}\beta(\ln t)\right)}\,. \tag{4.29}$$

Proof. — On restricting $\chi \in \mathscr{F}$ to the function $\chi : T \to \mathbb{C}$, it follows from Cor. 4.7 that $\dot{D}(I_\ell)\chi(t) = 0$ for all $\ell \in \mathbb{N}$. By Thm. 3.9 the radial part $\dot{D}(I_\ell)$ agrees with the differential operator $J^{-1/2} D_\ell \circ J^{1/2}$ modulo lower-order terms, $J^{-1/2} Q_{\ell-1} \circ J^{1/2}$. Since the lower-order operators $Q_{\ell-1}$ are themselves expressed as polynomials in the D_k and all constant terms $Q_{\ell-1}(\mathbf{0},\mathbf{0})$ vanish (Cor. 3.11), the system of equations $\dot{D}(I_\ell)\chi = 0$ is equivalent to the system $J^{-1/2} D_\ell (J^{1/2}\chi) = 0$ ($\ell \in \mathbb{N}$).

The expression for the function J follows from Thm. 3.9 by analytically continuing from the compact torus $\mathrm{U}_1^n \times \mathrm{U}_1^n$ to the Abelian semigroup T. ☐

The only input we required for Prop. 4.11 was the system of differential equations $D(I_\ell)\chi = 0$. Since this system is available also under the conditions of Cor. 4.5, the same conclusion holds true in that modified context. We record this fact for later use:

Corollary 4.12. — *Let $\tilde{\chi}$ be the character of an irreducible $\mathfrak{gl}_{n|n}$-representation on a finite-dimensional $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$. If $\dim(V_0) \neq \dim(V_1)$ then $J^{-1/2} D_\ell (J^{1/2}\tilde{\chi}) = 0$ for all $\ell \in \mathbb{N}$.*

Remark. — To obtain the statement of Prop. 4.11 by reference to standard text, it was necessary to go to the expense of extending χ to a section of $\mathscr{F} = \Gamma(M, \mathbb{C} \otimes \wedge F^*)$. With the system of differential equations $J^{-1/2} D_\ell (J^{1/2} \chi) = 0$ now established, the objects M and $\mathscr{F}$ have served their purpose, and we restrict all further considerations to the torus function $\chi : T \to \mathbb{C}$.

4.7. Proof of the main theorem. — We have now accumulated enough information about the character $t \mapsto \chi(t)$ to prove Thm. 1.1. Let us summarize what we know.

A. χ is a W-invariant analytic function on the Abelian semigroup T of Cor. 4.9.
B. χ has an expansion in terms of weights (Cor. 2.13) with coefficients in the limited range given by Prop. 2.12.
C. χ by Prop. 4.11 obeys the differential equations $J^{-1/2} D_\ell (J^{1/2} \chi) = 0$ $(\ell \in \mathbb{N})$.

The proof of Thm. 1.1 is done in two steps: we first show that the conditions (A-C) admit at most one solution χ; afterwards we will write down the solution and verify that it has the required properties. Of course, once Thm. 1.1 has been proved for $t \in T$, the result immediately extends to the complex torus by analytic continuation.

4.7.1. Uniqueness of the solution. — We shall prove that the coefficients of the weight expansion of χ (property B) are completely determined by W-invariance (property A) and the system of differential equations (property C).

Our first step is to establish a good way of representing the square root function $J^{1/2}$. Let $\delta : \mathfrak{h} \to \mathbb{C}$ be half the sum (in the $\mathbb{Z}_2$-graded sense) of positive roots:

$$\delta = \tfrac{1}{2} \textstyle\sum_{\alpha \in \Delta_0^+} \alpha - \tfrac{1}{2} \sum_{\beta \in \Delta_1^+} \beta \, , \tag{4.30}$$

and choose a square root of the expression (4.29) for J :

$$J^{1/2} = \mathrm{e}^{\delta} \, \frac{\prod_{\alpha \in \Delta_0^+} \frac{1}{2}(1 - \mathrm{e}^{-\alpha})}{\prod_{\beta \in \Delta_1^+} \frac{1}{2}(1 - \mathrm{e}^{-\beta})} \circ \ln \, . \tag{4.31}$$

Notice that while the function $t \mapsto \mathrm{e}^{\delta(\ln t)}$ is double-valued, the square root ambiguity cancels in the differential operator $J^{-1/2} D_\ell \circ J^{1/2}$.

The following subsystem of $r = 2(p+q) - 1$ positive roots $\{\sigma_1, \ldots, \sigma_r\} \subset \Delta^+$:

$$\begin{aligned} &\phi_j - \phi_{j-1} \; (j = 2, \ldots, p) \, , \quad \mathrm{i}\psi_1 - \phi_p \, , \quad \mathrm{i}\psi_k - \mathrm{i}\psi_{k-1} \; (k = 2, \ldots, p+q) \, , \\ &\phi_{p+1} - \mathrm{i}\psi_{p+q} \, , \quad \phi_l - \phi_{l-1} \; (l = p+2, \ldots, p+q) \, , \end{aligned} \tag{4.32}$$

is a system of simple roots, i.e., every positive root α is uniquely expressed as a linear combination $\alpha = \sum_i a_i \sigma_i$ with non-negative integers a_i.

Lemma 4.13. — *The function* $\mathrm{e}^{-\delta \circ \ln} J^{1/2} : T \to \mathbb{C}$ *has an absolutely convergent expansion in terms of simple roots* $\sigma_1, \ldots, \sigma_r$:

$$\mathrm{e}^{-\delta \circ \ln} J^{1/2} = \sum A_b \, \mathrm{e}^{-\sum_{i=1}^r b_i \sigma_i \circ \ln} \quad (A_b \in \mathbb{C} \, , \; A_{b=\mathbf{0}} \neq 0) \, ,$$

where the sum runs over sets $b \equiv \{b_1, \ldots, b_r\}$ *composed of non-negative integers* b_i.

Proof. — Expand the factors $1-\mathrm{e}^{-\beta\circ\ln}$ in the denominator of (4.31) as geometric series. If $\ln t \in \mathfrak{t}$ one has $|\mathrm{e}^{-\beta(\ln t)}| = \mathrm{e}^{-\Re\mathfrak{e}\,\beta(\ln t)} < 1$ for all $\beta \in \Delta_1^+$. Therefore, these series converge absolutely on T, and after expressing all of the roots $\alpha, \beta \in \Delta^+$ in terms of the simple roots σ_i, the series can be reordered to the stated form, with the resulting series still being absolutely convergent. □

To draw a conclusion from the differential equations $J^{-1/2}D_\ell(J^{1/2}\chi) = 0$, we need some understanding of the joint kernel of the set of differential operators D_ℓ. Consider the exponential function $f = \mathrm{e}^{\sum_k(\mathrm{i}m_k\psi_k - n_k\phi_k)}$ for some set of numbers m_k, n_k.

Lemma 4.14. — *The function* $f = \mathrm{e}^{\sum_k(\mathrm{i}m_k\psi_k - n_k\phi_k)\circ\ln}$ *is annihilated by all of the operators* D_ℓ ($\ell \in \mathbb{N}$) *if and only if* $\{m_1, \ldots, m_{p+q}\}$ *is the same set as* $\{n_1, \ldots, n_{p+q}\}$.

Proof. — f is a joint eigenfunction:

$$\mathrm{e}^{-\sum_k(\mathrm{i}m_k\psi_k - n_k\phi_k)\circ\ln} D_\ell\big(\mathrm{e}^{\sum_k(\mathrm{i}m_k\psi_k - n_k\phi_k)\circ\ln}\big) = \mathrm{i}^\ell \textstyle\sum_{k=1}^{p+q}(m_k^\ell - n_k^\ell) ,$$

and the eigenvalue vanishes for all $\ell \in \mathbb{N}$ if and only if $n_k = \pi(m_k)$ $(k = 1, \ldots, p+q)$ for some permutation $\pi \in \mathrm{S}_{p+q}$. □

If we were dealing with a non-degenerate representation, the joint eigenspace of the operators D_ℓ (corresponding to the given set of Casimir eigenvalues) would have a dimension no larger than the order of the Weyl group. In contrast, the joint kernel of the operators D_ℓ is seen to be huge! One might therefore think that our character χ is hopelessly underdetermined by the system of equations $J^{-1/2}D_\ell(J^{1/2}\chi) = 0$. However, the situation will be saved by Prop. 2.12, which says that $\gamma = \sum(\mathrm{i}m_k\psi_k - n_k\phi_k)$ is a weight of $\mathscr{V}_\lambda$ only if the coefficients m_k and n_k are integers in the range

$$n_j \le 0 \le m_k \le N \le n_l \quad (1 \le j \le p < l \le p+q) . \tag{4.33}$$

Recall now from Cor. 2.13 the weight expansion of the character χ, which involved unknown multiplicities for the weights $\gamma \in \Gamma_\lambda$. For present purposes, let this expansion be written as

$$\chi(t) = \textstyle\sum_{\gamma\in\Gamma_\lambda} B_\gamma\, \mathrm{e}^{\gamma(\ln t)} , \tag{4.34}$$

where $B_\gamma = (-1)^{|\gamma|}\dim(V_\gamma)$ are the unknowns. Note $B_{\lambda_N} = 1$, since the highest-weight space is one-dimensional. W-invariance then implies $B_{w(\lambda_N)} = 1$ for all $w \in W$.

Every weight $\gamma \in \Gamma_\lambda$ is uniquely represented in terms of the highest weight λ_N and the simple roots (4.32) as $\gamma = \lambda_N - \sum_{i=1}^r c_i\sigma_i$ with non-negative integers c_i. We refer to the uniquely determined number $\sum_i c_i \ge 0$ as the *degree* of the weight γ.

Lemma 4.15. — *If the system of differential equations* $J^{-1/2}D_\ell(J^{1/2}\chi) = 0$ ($\ell \in \mathbb{N}$) *admits a* W*-invariant solution of the form (4.34) with* $B_{\lambda_N} = 1$ *and weights* γ *that have coefficients in the range (4.33), then this solution is unique.*

Proof. — By multiplication with the function $\mathrm{e}^{-\delta\circ\ln}J^{1/2}$, write the system of differential equations in the equivalent form

$$\forall \ell \in \mathbb{N}: \quad 0 = \mathrm{e}^{-\delta\circ\ln} D_\ell(J^{1/2}\chi) ,$$

and use for $J^{1/2}$ the absolutely convergent expansion given by Lem. 4.13. By Cor. 2.13 the domain of absolute convergence of the weight expansion (4.34) of χ contains T. Thus one may insert (4.34) to obtain

$$\forall \ell \in \mathbb{N}: \quad 0 = \sum A_b \sum_\gamma B_\gamma \mathrm{e}^{-\delta \circ \ln} D_\ell \left(\mathrm{e}^{(\delta + \gamma - \sum b_i \sigma_i) \circ \ln} \right) , \tag{4.35}$$

where the b-sum is over $b_i \geq 0$. The statement to be proved now is that the solution of these equations for the unknowns B_γ is unique (assuming that a solution exists).

Interchanging the sum over $\gamma \in \Gamma_\lambda$ with the sum over the b_i's and making the substitution $\gamma \to \gamma + \sum b_i \sigma_i$, one reorganizes the system (4.35) as

$$\forall \ell \in \mathbb{N}: \quad 0 = \sum_\gamma \left(\sum_{\sum b_i \geq 0} A_b B_{\gamma + \sum b_i \sigma_i} \right) \mathrm{e}^{-\delta \circ \ln} D_\ell \left(\mathrm{e}^{(\delta + \gamma) \circ \ln} \right) . \tag{4.36}$$

By standard reasoning the functions $\mathrm{e}^{(\delta+\gamma)\circ \ln}$ which appear in the outer sum, all are linearly independent of each other. They are eigenfunctions of the operators D_ℓ. The inner sum is a finite sum because the degree of the summation variable $\gamma' \equiv \gamma + \sum b_i \sigma_i$ decreases with increasing values of the coefficients b_i and the multiplicity $B_{\gamma'}$ vanishes when this degree is less than the degree of the highest weight λ_N.

The plan then is to solve the system of equations (4.35) for the unknowns B_γ recursively, by starting from $B_{\lambda_N} = 1$ and proceeding in ascending order of the total degree of $\gamma \in \Gamma_\lambda$. An unknown multiplicity B_γ will be uniquely determined by the recursive scheme (4.36) if the function $\mathrm{e}^{(\delta+\gamma)\circ\ln}$ does not lie in the joint kernel of the differential operators D_ℓ. Indeed, in that case one deduces from (4.36) the equation

$$0 = A_{\mathbf{0}} B_\gamma + \sum_{\sum b_i \geq 1} A_b \, B_{\gamma + \sum b_i \sigma_i} , \tag{4.37}$$

and since $A_{b=\mathbf{0}} \neq 0$, this determines B_γ from the already known $B_{\tilde\gamma}$ for the weights $\tilde\gamma = \gamma + \sum b_i \sigma_i$ with $\sum b_i \geq 1$, which are of lower degree than the weight γ in question.

It remains to show that there is no indeterminacy from solutions of $D_\ell(\mathrm{e}^{(\delta+\gamma)\circ\ln}) = 0$ for $\gamma \in \Gamma_\lambda$. The half sum of positive roots is readily computed to be

$$\delta = \sum_{k=1}^{p+q} (k - p - \tfrac{1}{2}) \mathrm{i}\psi_k + \sum_{j=1}^{p} (j - \tfrac{1}{2}) \phi_j + \sum_{l=p+1}^{p+q} (l - p - q - \tfrac{1}{2}) \phi_l ,$$

and adding $\gamma \in \Gamma_\lambda$ one gets from (4.33) that $\delta + \gamma = \sum_k (\mathrm{i}\tilde m_k \psi_k - \tilde n_k \phi_k)$ with

$$\begin{aligned} \tilde n_{p-j+1} \quad &\leq \quad j - p - \tfrac{1}{2} \quad (j = 1, \ldots, p) , \\ k - p - \tfrac{1}{2} \quad &\leq \quad \tilde m_k \ \leq \ N + k - p - \tfrac{1}{2} \quad (k = 1, \ldots, p+q) , \\ N + l - p - \tfrac{1}{2} \quad &\leq \quad \tilde n_{2p+q-l+1} \quad (l = p+1, \ldots, p+q) . \end{aligned}$$

By a simple induction one proves that, given these inequalities, there exists just one way to arrange for the $\tilde m_k$ to be a permutation of the $\tilde n_k$. This unique arrangement occurs when the inequalities are saturated as $\tilde m_j = \tilde n_{p-j+1} = j - p + \frac{1}{2}$ (for $j = 1, \ldots, p$) and $\tilde m_l = \tilde n_{2p+q-l+1} = N + l - p - \frac{1}{2}$ (for $l = p+1, \ldots, p+q$). Returning from $\gamma + \delta$ to γ we see that these values of the coefficients $\tilde m_k$ and $\tilde n_k$ correspond to the coefficients m_k and n_k of the highest weight λ_N.

By Lem. 4.15 it follows that $\gamma = \lambda_N$ is the only solution (for $\gamma \in \Gamma_\lambda$) of the system of equations $D_\ell(\mathrm{e}^{(\delta+\gamma)\circ\ln}) = 0$ for all $\ell \in \mathbb{N}$. Thus there is no indeterminacy and the weight expansion (4.34) for χ has now been shown to be uniquely determined. □

4.7.2. Explicit solution. — As a preparation for writing down and establishing a solution of the conditions (A-C), we trisect the system of positive roots Δ^+.

Recall from Sect. 2.7 the decomposition

$$\mathrm{Hom}(U^+,U^-) \oplus \mathfrak{gl}(U^+) \oplus \mathfrak{gl}(U^-) \oplus \mathrm{Hom}(U^-,U^+) \hookrightarrow \mathfrak{gl}(U) .$$

Δ_λ^+ was defined in (2.39) as the set of positive roots α with the property that the root space $\mathfrak{g}^{-\alpha}$ lies in the space of degree-increasing operators $\mathfrak{g}^{(2)} = \mathrm{Hom}(U^-,U^+) \subset \mathfrak{n}^-$; these are the roots listed in (2.40). The complementary set $\Delta^+ \setminus \Delta_\lambda^+$ splits into two subsets associated with $\mathfrak{gl}(U^+) = \mathfrak{gl}_{p|p}$ and $\mathfrak{gl}(U^-) = \mathfrak{gl}_{q|q}$:

$$\Delta_p^+ = \{\alpha \in \Delta^+ \mid \mathfrak{g}^\alpha \subset \mathfrak{gl}(U^+)\} , \quad \Delta_q^+ = \{\alpha \in \Delta^+ \mid \mathfrak{g}^\alpha \subset \mathfrak{gl}(U^-)\} . \tag{4.38}$$

The first set Δ_p^+ is made from the sequence

$$\phi_1 , \ldots , \phi_p , \mathrm{i}\psi_1 , \ldots , \mathrm{i}\psi_p ,$$

by taking differences $x - y$ with x occurring later in the sequence than y, just like in (2.32). The other set, Δ_q^+, is obtained by doing the same with the sequence

$$\mathrm{i}\psi_{p+1} , \ldots , \mathrm{i}\psi_{p+q} , \phi_{p+1} , \ldots , \phi_{p+q} .$$

Thus we can express the system of positive roots Δ^+ as a disjoint union of three sets:

$$\Delta^+ = \Delta_p^+ \cup \Delta_q^+ \cup \Delta_\lambda^+ . \tag{4.39}$$

Let the corresponding factorization of the function J of (4.29) be written as

$$J = J_p J_q Z^{-2} . \tag{4.40}$$

If $\Delta_p^+ = \Delta_{p,0}^+ \cup \Delta_{p,1}^+$ is the decomposition into even and odd roots, the first factor is

$$J_p = \frac{\prod_{\alpha \in \Delta_{p,0}^+} \sinh^2(\frac{1}{2}\alpha)}{\prod_{\beta \in \Delta_{p,1}^+} \sinh^2(\frac{1}{2}\beta)} \circ \ln , \tag{4.41}$$

and the second factor, J_q, is analogous. The third factor is the inverse square of

$$Z = \frac{\prod_{\beta \in \Delta_{\lambda,1}^+} \sinh(\frac{1}{2}\beta)}{\prod_{\alpha \in \Delta_{\lambda,0}^+} \sinh(\frac{1}{2}\alpha)} \circ \ln = \prod_{j=1}^{p} \prod_{l=p+1}^{p+q} \frac{(1-\mathrm{e}^{\mathrm{i}\psi_j - \phi_l})(1-\mathrm{e}^{\phi_j - \mathrm{i}\psi_l})}{(1-\mathrm{e}^{\mathrm{i}\psi_j - \mathrm{i}\psi_l})(1-\mathrm{e}^{\phi_j - \phi_l})} \circ \ln . \tag{4.42}$$

All of these functions exist on the subset of *regular elements* $T' \subset T$.

Recall from Sect. 4.5 that Weyl group elements $w \in W$ act primarily on the torus T by $t \mapsto w \cdot t$; this induces an action on functions $f : T \to \mathbb{C}$ by $w(f)(t) = f(w^{-1} \cdot t)$. We also have induced actions on the tangent vector space $\mathfrak{h}_\mathbb{R} \supset \mathfrak{t}$ and on linear functions on $\mathfrak{h}_\mathbb{R}$. Let $W_\lambda \subset W$ be the subgroup that stabilizes the highest weight λ_N.

If $f : T \to \mathbb{C}$ is W_λ-invariant, define the W-symmetrized function $S_W f$ by

$$S_W f(t) := \sum_{[w]\in W/W_\lambda} f(w^{-1}\cdot t) \; .$$

Note that since the set Δ_λ^+ is invariant under the induced action of W_λ, so is the function Z constructed from it.

Lemma 4.16. — *The function $\chi : T' \to \mathbb{C}$ defined by*

$$\chi(t) = S_W \chi_\lambda(t) \; , \quad \chi_\lambda(t) = \mathrm{e}^{\lambda_N(\ln t)} Z(t) \; ,$$

is a solution of the system of differential equations $J^{-1/2} D_\ell (J^{1/2}\chi) = 0$ (property C).

Proof. — Take the square root $J^{1/2} = J_p^{1/2} J_q^{1/2} Z^{-1}$. Then notice that the differential operators D_ℓ, which are sums of powers of partial derivatives, split according to $\mathfrak{g}^{(0)} = \mathfrak{gl}(U^+) \oplus \mathfrak{gl}(U^-) = \mathfrak{gl}_{p|p} \oplus \mathfrak{gl}_{q|q}$ as $D_\ell = D_\ell^+ + D_\ell^-$, so that

$$J_p^{-1/2} J_q^{-1/2} D_\ell \circ J_p^{1/2} J_q^{1/2} = J_p^{-1/2} D_\ell^+ \circ J_p^{1/2} + J_q^{-1/2} D_\ell^- \circ J_q^{1/2} \; .$$

Since the D_ℓ are W-invariant and commute with the symmetrizer S_W, one has

$$J^{-1/2} D_\ell (J^{1/2}\chi) = J^{-1/2} S_W D_\ell (J_p^{1/2} J_q^{1/2} \mathrm{e}^{\lambda_N \circ \ln}) \; .$$

With these identities in place, in order to establish property C it is sufficient to show that the function $t \mapsto \mathrm{e}^{\lambda_N(\ln t)}$ obeys the system of differential equations

$$J_p^{-1/2} D_\ell^+ (J_p^{1/2} \mathrm{e}^{\lambda_N\circ\ln}) = J_q^{-1/2} D_\ell^- (J_q^{1/2} \mathrm{e}^{\lambda_N\circ\ln}) = 0 \quad (\ell \in \mathbb{N}) \; .$$

But these follow in turn from Cor. 4.12. Indeed, $\mathrm{e}^{\lambda_N\circ\ln}$ with $\lambda_N = N \sum_{l=p+1}^{p+q} (\mathrm{i}\psi_l - \phi_l)$ is the primitive character of $\mathrm{GL}_{p|p} \times \mathrm{GL}_{q|q}$ which is given by the trivial representation of $\mathrm{GL}_{p|p}$ and the SDet^{-N}-representation of $\mathrm{GL}_{q|q}$. Both of these representations are one-dimensional and thus of unequal even and odd dimension ($\dim V_0 = 1$ and $\dim V_1 = 0$). Hence Cor. 4.12 applies, and property C is now established. ☐

Next, we address the question whether the function χ defined in Lem. 4.16 possesses the property A of our list. For that, we write the function Z of (4.42) in the form

$$Z = \prod_{j=1}^{p} \prod_{l=p+1}^{p+q} \frac{\sinh\left(\frac{1}{2}(\phi_l - \mathrm{i}\psi_j)\right) \sinh\left(\frac{1}{2}(\mathrm{i}\psi_l - \phi_j)\right)}{\sinh\left(\frac{1}{2}(\mathrm{i}\psi_l - \mathrm{i}\psi_j)\right) \sinh\left(\frac{1}{2}(\phi_l - \phi_j)\right)} \circ \ln \; .$$

Clearly, in the domain T, where $\phi_j < 0 < \phi_l$, the function $\mathrm{e}^{\lambda_N\circ\ln} Z$ and all of its W-translates are analytic in the ϕ-variables. Z, however, has poles at $\psi_l = \psi_j \pmod{2\pi\mathrm{i}}$.

The issue at stake now is what happens with these singularities under symmetrization by the Weyl group W. To answer that, let us adopt the convention that $j, j', j'' \in \{1, \ldots, p\}$ and $l, l', l'' \in \{p+1, \ldots, p+q\}$. Then, fixing any pair j, l, let $T_{jl} \subset T$ be the zero locus of the factor $\sin\left(\frac{1}{2}(\psi_l - \psi_j)\right)$ of the denominator of Z, excluding the zero loci of other factors. If $w \in W$, there exist two types of outcome for the action of w on the corresponding functions ψ_j, ψ_l. Either we have one of the two situations

$$w(\psi_j) = \psi_{j'} \; , \; w(\psi_l) = \psi_{l'} \; , \quad \text{or} \quad w(\psi_j) = \psi_{l'} \; , \; w(\psi_l) = \psi_{j'} \; ,$$

for some pair j', l', in which case the singular factor $\sin\left(\frac{1}{2}(\psi_l - \psi_j)\right)$ still occurs in the denominator of the transformed function $w(Z)$; or else we have

$$w(\psi_j) = \psi_{j'}\ ,\ w(\psi_l) = \psi_{j''}\ , \quad \text{or} \quad w(\psi_j) = \psi_{l'}\ ,\ w(\psi_l) = \psi_{l''}\ ,$$

in which case the singular factor is absent and $w(Z)$ is obviously well-behaved on T_{jl}. Thus we may organize the sum over W-translates for χ in two parts:

$$\chi = \frac{A^{(jl)}}{\sin\left(\frac{1}{2}(\psi_l - \psi_j) \circ \ln\right)} + B^{(jl)}\ , \tag{4.43}$$

where $B^{(jl)}$ is the sum of all terms that are non-singular on T_{jl}.

Now observe that there exists an element $w_{jl} \in W$ with the property that its action interchanges ψ_j with ψ_l and leaves all other ψ_k the same. This w_{jl} clearly preserves the organization of χ into singular and non-singular terms. Since χ is W-invariant, we have $w_{jl}(B^{(jl)}) = B^{(jl)}$ and, since $\sin\left(\frac{1}{2}(\psi_l - \psi_j)\right)$ is w_{jl}-odd, it follows that

$$w_{jl}\,(A^{(jl)}) = -A^{(jl)}\ .$$

Given the expression for Z, this means that $A^{(jl)}$ vanishes at least linearly on T_{jl}. Thus the apparent pole is cancelled and both summands in (4.43) remain finite there.

Simultaneous poles from several factors of the denominator $\prod \sin\left(\frac{1}{2}(\psi_l - \psi_j)\right)$ are handled in the same way. Thus we conclude that the W-invariant function $\chi : T' \to \mathbb{C}$ defined in Lem. 4.16 continues to an analytic function $\chi : T \to \mathbb{C}$.

Proposition 4.17. — *The analytic function $\chi : T \to \mathbb{C}$ which is given by $\chi = S_W \chi_\lambda$ and $\chi_\lambda(t) = \mathrm{e}^{\lambda_N(\ln t)} Z(t)$ is a solution to the problem posed by the conditions (A-C).*

Proof. — It remains to prove that the proposed solution χ has property B, i.e., if χ is expanded as $\chi(t) = \sum_\gamma B_\gamma \mathrm{e}^{\gamma(\ln t)}$, then all of the weights $\gamma = \sum_k (\mathrm{i} m_k \psi_k - n_k \phi_k)$ occurring in this expansion have integer coefficients in the range

$$n_j \le 0 \le m_k \le N \le n_l\ ,$$

and $B_{\lambda_N} = 1$. We still refer to the linear functions γ as "weights", even though in the present context we must not use the fact that they are the weights of a representation.

The first step is to expand the function Z by exponentials. For that purpose, we add some real constants to the imaginary parameters $\mathrm{i}\psi_k$ and define a positive chamber $\tilde{\mathfrak{t}}^+ \subset \mathfrak{h}$ of real dimension $2(p+q)$ by

$$\tilde{\mathfrak{t}}^+ : \quad \phi_1 < \ldots < \phi_p < \mathfrak{Re}\,(\mathrm{i}\psi_1) < \ldots < \mathfrak{Re}\,(\mathrm{i}\psi_{p+q}) < \phi_{p+1} < \ldots < \phi_{p+q}\ .$$

Note that if $H \in \tilde{\mathfrak{t}}^+$ then $\mathfrak{Re}\,\alpha(H) > 0$ for all $\alpha \in \Delta^+$. Put $\tilde{T}_+ := \exp \tilde{\mathfrak{t}}^+$.

Now rewrite Z from (4.42) as

$$Z = \frac{\prod_{\beta \in \Delta^+_{\lambda,1}} (1 - \mathrm{e}^{-\beta})}{\prod_{\alpha \in \Delta^+_{\lambda,0}} (1 - \mathrm{e}^{-\alpha})} \circ \ln\ .$$

Since $\Delta^+_{\lambda,0} \subset \Delta^+$, and $|\mathrm{e}^{-\alpha(H)}| < 1$ for $H \in \tilde{\mathfrak{t}}^+$ and $\alpha \in \Delta^+$, each factor $(1 - \mathrm{e}^{-\alpha \circ \ln})^{-1}$ of $\chi_\lambda = \mathrm{e}^{\lambda_N \circ \ln} Z$ can be expanded on $\tilde{T}_+$ as a convergent geometric series.

A slightly modified expansion procedure works for all of the W-translates $w(\chi_\lambda)$: if a root $w(\alpha)$ for $\alpha \in \Delta^+_{\lambda,0}$ is positive, we expand as before; if it is not, we first rearrange

$$(1 - \mathrm{e}^{-w(\alpha)})^{-1} = -\mathrm{e}^{w(\alpha)}(1 - \mathrm{e}^{w(\alpha)})^{-1} \,,$$

and then expand. In this way, we produce for $\chi = S_W \chi_\lambda$ an expansion

$$\chi(t) = \sum\nolimits_\gamma B_\gamma \mathrm{e}^{\gamma(\ln t)} \,, \quad \gamma = \sum\nolimits_k (\mathrm{i} m_k \psi_k - n_k \phi_k) \,, \tag{4.44}$$

with integers m_k and n_k, which is absolutely convergent on $\tilde{T}_+$.

In the case of the coefficients n_k, since the action of the Weyl group W preserves the inequalities $\phi_j < 0 < \phi_l$, the desired restriction on the range is obvious from $\lambda_N = N \sum (\mathrm{i}\psi_l - \phi_l)$ and the system of λ_N-positive roots $\Delta^+_{\lambda,0}$. What needs to be proved, though, is the inequality for the coefficients m_k :

$$0 \le m_k \le N \quad (1 \le k \le p+q) \,. \tag{4.45}$$

To that end, notice first of all that the set of $(p+q)$-tuples $(m_1, \ldots, m_{p+q})$ occurring in (4.44) certainly is a *bounded* subset, say D_{λ_N}, of $\mathbb{Z}^{p+q}$. Indeed, if this was not so, then there would be an immediate contradiction with the known analyticity of χ when any one of the factors $\mathrm{e}^{\mathrm{i}\psi_j - \mathrm{i}\psi_l}$ passes through unity. (In other words, our expansion by exponentials must remain convergent when we let $\tilde{t} \in \tilde{T}_+$ tend to $t \in T$.)

Although sharp bounds on D_{λ_N} are not easy to establish directly, the following statement is immediate: D_{λ_N} lies in the intersection of the sets of $m_1 \ge 0$ and $m_{p+q} \le N$. Indeed, writing the proposed solution χ as

$$\chi = \sum_{[w] \in W/W_\lambda} \mathrm{e}^{w(\lambda_N)} \prod_{j=1}^{p} \prod_{l=p+1}^{p+q} \frac{(1 - \mathrm{e}^{\phi_j - \mathrm{i}w(\psi_l)})(1 - \mathrm{e}^{\mathrm{i}w(\psi_j) - \phi_l})}{(1 - \mathrm{e}^{\mathrm{i}w(\psi_j) - \mathrm{i}w(\psi_l)})(1 - \mathrm{e}^{\phi_j - \phi_l})} \circ \ln \,, \tag{4.46}$$

and inspecting the terms in the sum over W-translates where $w(\psi_l) \neq \psi_1$ for all $l = p+1, \ldots, p+q$, it is clear that negative powers of $\mathrm{e}^{\mathrm{i}\psi_1}$ never arise from such terms in our expansion. For the summands where $w(\psi_l) = \psi_1$ does occur for some l, we write

$$\prod_j \frac{1 - \mathrm{e}^{\phi_j - \mathrm{i}w(\psi_l)}}{1 - \mathrm{e}^{\mathrm{i}w(\psi_j) - \mathrm{i}w(\psi_l)}} = \prod_j \mathrm{e}^{-\mathrm{i}w(\psi_j)} \frac{\mathrm{e}^{\phi_j} - \mathrm{e}^{\mathrm{i}\psi_1}}{1 - \mathrm{e}^{\mathrm{i}\psi_1 - \mathrm{i}w(\psi_j)}} \,,$$

and since the roots $\mathrm{i}w(\psi_j) - \mathrm{i}\psi_1$ are in Δ^+, the new denominators are ready for expansion, and we see that the dependence on ψ_1 is always $\mathrm{e}^{\mathrm{i}m_1\psi_1}$ with $m_1 \ge 0$.

Similarly, the inequality $m_{p+q} \le N$ is obvious, with the exception of the terms of the $[w]$-sum where $w(\psi_j) = \psi_{p+q}$ for some $j \in \{1, \ldots, p\}$. But in those cases we write

$$\prod_l \frac{1 - \mathrm{e}^{\mathrm{i}w(\psi_j) - \phi_l}}{1 - \mathrm{e}^{\mathrm{i}w(\psi_j) - \mathrm{i}w(\psi_l)}} = \prod_l \mathrm{e}^{\mathrm{i}w(\psi_l)} \frac{\mathrm{e}^{-\phi_l} - \mathrm{e}^{-\mathrm{i}\psi_{p+q}}}{1 - \mathrm{e}^{\mathrm{i}w(\psi_l) - \mathrm{i}\psi_{p+q}}} \,,$$

where all denominators are again of the form $(1 - \mathrm{e}^{-\alpha})^{-1}$ with $\alpha \in \Delta^+$. Because e^{λ_N} contains a factor $\mathrm{e}^{\mathrm{i}N\psi_{p+q}}$ we cannot infer $m_{p+q} \le 0$, but it does follow that $m_{p+q} \le N$.

Thus we have established that D_{λ_N} does not intersect either of the sets $m_1 < 0$ or $m_{p+q} > N$. This of course still leaves a set much bigger than the desired one of (4.45).

We now *repeat*, however, the whole procedure for *another* choice of highest weight $w(\lambda_N) \neq \lambda_N$ ($w \in W$). Thus, replacing the system of positive roots Δ^+ by a system of positive roots $w(\Delta^+)$, and the domain $\tilde{T}_+$ by a *transformed* domain $w(\tilde{T}_+)$, we produce another expansion for χ by exponentials where the $(p+q)$-tupels of integers $(m_1, \ldots, m_{p+q})$ in (4.44) are another set $D_{w(\lambda_N)} \subset \mathbb{Z}^{p+q}$. There exists such a choice of highest weight $w_{w(\lambda_N)}$ that the roles formerly played by ψ_1 and ψ_{p+q} are now played by ψ_2 resp. ψ_{p+q-1}. With this particular choice, by the same argument as before, the set $D_{w(\lambda_N)}$ has zero intersection with $m_2 < 0$ and with $m_{p+q-1} > N$.

A priori, the second expansion could be different from the first one. However, since the two expansions represent the same analytic function and both sets D_{λ_N} and $D_{w(\lambda_N)}$ are bounded, they must actually coincide: $D_{\lambda_N} = D_{w(\lambda_N)}$. It follows that the integers m_k of our two identical expansions have zero intersection with each of the sets $m_1 < 0$, $m_2 < 0$, $m_{p+q-1} > N$, and $m_{p+q} > N$.

By continuing in this fashion, one eventually cuts down the range of the integers m_k to that of (4.45). It is easy to see that $B_{\lambda_N} = 1$. This completes the proof. □

From (4.46) our solution to the problem posed by the conditions (A-C) coincides with the one stated in Thm. 1.1. Since the solution is unique by Lem. 4.15, the proof of that theorem is now finished but for the need to prove Prop. 4.4.

4.8. Proof of Cor. 1.2. — The claim made in Cor. 1.2 is that

$$\int_{\mathrm{U}_N} \frac{\prod_{j=1}^{p} \mathrm{Det}(\mathrm{Id}_N - \mathrm{e}^{\mathrm{i}\psi_j} u) \prod_{l=p+1}^{p+q} \mathrm{Det}(\mathrm{Id}_N - \mathrm{e}^{-\mathrm{i}\psi_l} \bar{u})}{\prod_{j'=1}^{p'} \mathrm{Det}(\mathrm{Id}_N - \mathrm{e}^{\phi_{j'}} u) \prod_{l'=p'+1}^{p'+q'} \mathrm{Det}(\mathrm{Id}_N - \mathrm{e}^{-\phi_{l'}} \bar{u})} du \tag{4.47}$$
$$= \frac{1}{p!\,q!} \sum_{w \in \mathrm{S}_{p+q}} \prod_{k=p+1}^{p+q} \frac{\mathrm{e}^{N\mathrm{i}w(\psi_k)}}{\mathrm{e}^{N\mathrm{i}\psi_k}} \times \frac{\prod_{j',l}(1-\mathrm{e}^{\phi_{j'}-\mathrm{i}w(\psi_l)}) \prod_{j,l'}(1-\mathrm{e}^{\mathrm{i}w(\psi_j)-\phi_{l'}})}{\prod_{j,l}(1-\mathrm{e}^{\mathrm{i}w(\psi_j)-\mathrm{i}w(\psi_l)}) \prod_{j',l'}(1-\mathrm{e}^{\phi_{j'}-\phi_{l'}})}$$

holds as long as $p' \le p+N$ and $q' \le q+N$.

This formula is deduced from (4.46) by the method of induction as follows. To start the induction process, note that if $p = p'$ and $q = q'$, then (4.47) is just (4.46) restated by dividing both sides by e^{λ_N} and replacing the sum over cosets $\sum_{[w] \in W/W_\lambda}$ by the normalized sum over group elements $(p!\,q!)^{-1} \sum_{w \in \mathrm{S}_{p+q}}$.

Assume now that (4.47) holds true for some set of non-negative integers p, p', q, and q' in the specified range. Then by taking $\mathrm{e}^{-\mathrm{i}\psi_{p+q}}$ to zero, and assuming $q \ge 1$ and $q' \le (q-1)+N$, let us show that (4.47) still holds true when q is lowered to $q-1$.

For this purpose, divide the sum over permutations $w \in \mathrm{S}_{p+q}$ into two partial sums which are set apart by whether $w^{-1}(\psi_{p+q}) = \psi_j$ for $j \in \{1, \ldots, p\}$ or $w^{-1}(\psi_{p+q}) = \psi_l$ for $l \in \{p+1, \ldots, p+q\}$. In the former case the corresponding summand on the right-hand side of (4.47) goes to zero in the limit of $\mathrm{e}^{-\mathrm{i}\psi_{p+q}} \to 0$. Indeed, collecting all factors containing $w(\psi_j) \equiv \psi_{p+q}$ one gets

$$\frac{1}{\mathrm{e}^{N\mathrm{i}\psi_{p+q}}} \times \frac{\prod_{l'}(1-\mathrm{e}^{\mathrm{i}\psi_{p+q}-\phi_{l'}})}{\prod_{l}(1-\mathrm{e}^{\mathrm{i}\psi_{p+q}-w(\psi_l)})},$$

which scales as $\mathrm{e}^{-(N+q-q')\mathrm{i}\psi_{p+q}}$ when $\mathrm{e}^{-\mathrm{i}\psi_{p+q}}$ is small, and since $N+q-q' \geq 1$ by assumption, the contributions from such terms vanish when $\mathrm{e}^{-\mathrm{i}\psi_{p+q}}$ is set to zero.

Now turn to the second case, where $w^{-1}(\psi_{p+q}) = \psi_l$ for some $l \in \{p+1,\ldots,p+q\}$. There are $q \times (p+q-1)!$ such permutations. By explicit invariance under the second factor of $\mathrm{S}_p \times \mathrm{S}_q$ one may assume that $w(\psi_{p+q}) = \psi_{p+q}$ and make the replacement

$$q!^{-1}\sum\nolimits_{w\in \mathrm{S}_{p+q}} \to (q-1)!^{-1}\sum\nolimits_{w\in \mathrm{S}_{p+q-1}}$$

where $\mathrm{S}_{p+q-1} \subset \mathrm{S}_{p+q}$ is the subgroup of permutations that fix ψ_{p+q}. In all of these terms the product of factors containing the singular variable ψ_{p+q} is

$$\frac{\mathrm{e}^{N\,\mathrm{i}\psi_{p+q}}}{\mathrm{e}^{N\,\mathrm{i}\psi_{p+q}}} \times \frac{\prod_{j'}(1-\mathrm{e}^{\phi_{j'}-\mathrm{i}\psi_{p+q}})}{\prod_j(1-\mathrm{e}^{\mathrm{i}w(\psi_j)-\mathrm{i}\psi_{p+q}})} \;,$$

which goes to unity when $\mathrm{e}^{-\mathrm{i}\psi_{p+q}}$ is set to zero. The remaining factors on the right-hand side of (4.47) reproduce the desired answer for the lowered value $q-1$. On the left-hand side, setting $\mathrm{e}^{-\mathrm{i}\psi_{p+q}}$ to zero just removes the factor $\mathrm{Det}(\mathrm{Id}_N - \mathrm{e}^{-\mathrm{i}\psi_{p+q}}\,\overline{u})$. Altogether this shows that the induction step $q \to q-1$ is valid in the specified range.

In very much the same manner one establishes the validity of the induction step of $p \to p-1$ for $p \geq 1$ and $p' \leq (p-1)+N$.

Finally, it is clear that one can always lower p' and q' by sending $\mathrm{e}^{\phi_{j'}} \to 0$ resp. $\mathrm{e}^{-\phi_{l'}} \to 0$. In this way (4.47) follows in the full range $p' \leq p+N$ and $q' \leq q+N$.

5. Proof of the extended character formula

Key to our determination of the character χ was property C (the system of differential equations) which in turn resulted from the statement that $\chi : T \to \mathbb{C}$ extends to a radial section $\chi \in \mathscr{F}$ by Prop. 4.4. The proof of that proposition had to be postponed because of length; we return to it in the current section.

To prepare for the situation of our Howe dual pair $(\mathfrak{gl}_{n|n}, \mathrm{U}_N)$, we go back again to the basic setting (Sect. 2.5) of a $\mathbb{Z}_2$-graded vector space $V = V_1 \oplus V_0$ with Hermitian subspaces V_0 and V_1 and orthogonal decompositions $V_\tau = V_\tau^+ \oplus V_\tau^-$ $(\tau = 0,1)$. Introducing $s := \mathrm{Id}_{V_0^+} \oplus (-\mathrm{Id}_{V_0^-})$ we associate with the pseudo-unitary vector space $V_0^s \equiv (V_0\,, s)$ a non-compact Lie group $\mathrm{U}(V_0^s)$ and a semigroup $\mathrm{H}(V_0^s)$ by

$$\mathrm{U}(V_0^s) = \{g \in \mathrm{GL}(V_0) \mid g^\dagger s g = s\}\;, \quad \mathrm{H}(V_0^s) = \{g \in \mathrm{GL}(V_0) \mid g^\dagger s g < s\}\;. \tag{5.1}$$

Note that $\mathrm{H}(V_0^s)$ is open in $\mathrm{GL}(V_0)$, and that $\mathrm{U}(V_0^s) \subset \overline{\mathrm{H}(V_0^s)}$.

From Def. 2.7 recall the definition of the spinor-oscillator module $\mathscr{A}_V$ of V. We are now going to construct representations R and R' of the semigroup $\mathrm{GL}(V_1) \times \mathrm{H}(V_0^s)$ and the Lie group $\mathrm{U}(V_1) \times \mathrm{U}(V_0^s)$, which integrate the infinitesimal representation $R_* : \mathfrak{gl}(V) \to \mathfrak{gl}(\mathscr{A}_V)$ of Sect. 2.5. This, in combination with some superanalysis, will eventually lead to a proof of the character formula (4.24) of Prop. 4.4.

Our first step is to construct $R : \mathrm{GL}(V_1) \times \mathrm{H}(V_0^s) \to \mathrm{GL}(\mathscr{A}_V)$. For the first factor this is straightforward. We start from the canonical representation $\sigma : \mathrm{GL}(V_1) \to \mathrm{GL}(\wedge V_1)$ by $\sigma(g)(v \wedge v' \wedge \ldots) = (gv) \wedge (gv') \wedge \ldots$. Then, choosing for $d = \dim(V_1^-)$ a unitary

generator $\Omega \in \wedge^d(V_1^{-*})$ we have a non-degenerate pairing $\wedge^\bullet(V_1^-)\otimes\wedge^{d-\bullet}(V_1^-)\to\mathbb{C}$, $(a,b)\mapsto\Omega(a\wedge b)$ and hence an isomorphism $\tau:\wedge(V_1^-)\to\wedge(V_1^{-*})$. This gives a $\mathrm{GL}(V_1)$-representation on $\wedge(V_1^+)\otimes\wedge(V_1^{-*})$ and hence on $\mathscr{A}_V$. We denote it by $R_1 : \mathrm{GL}(V_1)\to\mathrm{GL}(\mathscr{A}_V)$. Since σ is unitary on $\mathrm{U}(V_1)$, so is $R_1' := R_1|_{\mathrm{U}(V_1)}$.

Recall that $g\in\mathrm{GL}(V_1)$ acts on $v+\varphi\in V_1\oplus V_1^*$ by $g(v+\varphi) = gv+\varphi\circ g^{-1}$. Recall also that in Def. 2.7 a Clifford action $w\mapsto\mathbf{c}(w)$ of $W_1 = V_1\oplus V_1^*$ on $\mathscr{A}_V$ was specified.

Lemma 5.1. — *The Clifford action* $\mathbf{c}: W_1\to\mathrm{End}(\mathscr{A}_V)$ *is* $\mathrm{GL}(V_1)$*-equivariant:*

$$R_1(g)^{-1}\mathbf{c}(w)R_1(g) = \mathbf{c}(g^{-1}w) \quad (\textit{for all } g\in\mathrm{GL}(V_1) \textit{ and } w\in W_1)\ .$$

Proof. — This is an immediate consequence of the formula $\mathrm{Ad}\circ\exp = \exp\circ\mathrm{ad}$ and the fact that $[R_*(X),\mathbf{c}(w)] = \mathbf{c}(X\cdot w)$ for $X\in\mathfrak{gl}(V_1)\hookrightarrow\mathfrak{gl}(V)$. □

5.1. More about the semigroup $\mathrm{H}(V_0^s)$. — We now turn to the sector of the vector space $V_0 = V_0^+\oplus V_0^-$ with pseudo-unitary structure given by the involution s. To construct a representation of the semigroup $\mathrm{H}(V_0^s)$ on the oscillator module $\mathrm{S}(V_0^+\oplus V_0^{-*})$ some serious analytic issues have to be addressed. The first prerequisite for that is Lem. 5.2, which provides some needed information about $\mathrm{H}(V_0^s)$. In the statement of that lemma and its proof, we succumb to the convenience of denoting the unit operator on V_0 simply by $\mathrm{Id}_{V_0}\equiv 1$. As usual, if $X\in\mathrm{End}(V_0)$, let $X\mapsto X^\dagger$ be the operation of taking the adjoint with respect to the Hermitian structure of V_0, and define $\mathfrak{Re}\,X = \frac{1}{2}(X+X^\dagger)$.

Lemma 5.2. — *If* $\zeta_s^+(V_0)$ *denotes the set of complex linear transformations*

$$\zeta_s^+(V_0) := \{X\in\mathrm{End}(V_0)\mid \mathfrak{Re}\,X>0\ , \mathrm{Det}(X+s)\neq 0\}\ ,$$

the (Cayley-type) rational mapping

$$a:\ \mathrm{H}(V_0^s)\to\mathrm{End}(V_0)\ ,\quad h\mapsto s\,\frac{1+h}{1-h} = a_h$$

is a bijection from the semigroup $\mathrm{H}(V_0^s)$ *to* $\zeta_s^+(V_0)$.

Proof. — $h\in\mathrm{H}(V_0^s)$ cannot have an eigenvalue at unity, for if there existed an eigenvector $\psi = h\psi$ then the defining inequality $h^\dagger sh<s$ of $\mathrm{H}(V_0^s)$ would imply

$$\langle\psi,s\psi\rangle = \langle h\psi,sh\psi\rangle = \langle\psi,h^\dagger sh\psi\rangle < \langle\psi,s\psi\rangle\ ,$$

which is a contradiction. Thus $1-h$ is regular, and the map a exists on $\mathrm{H}(V_0^s)$.

Adding and subtracting terms gives

$$0 < s-h^\dagger sh = \tfrac{1}{2}(1-h^\dagger)s(1+h)+\tfrac{1}{2}(1+h^\dagger)s(1-h)\ .$$

Multiplication by $(1-h^\dagger)^{-1}$ and $(1-h)^{-1}$ from the left resp. right then results in

$$0<\tfrac{1}{2}(a_h+a_h^\dagger)\ ,$$

which is the statement $\mathfrak{Re}\,a_h>0$. If $X=a_h$ then $X+s = 2s(1-h)^{-1}$ has an inverse and hence a non-vanishing determinant. Thus $h\mapsto a_h$ maps $\mathrm{H}(V_0^s)$ into $\zeta_s^+(V_0)$.

By the same token, the converse is also true: if $X \in \zeta_s^+(V_0)$ then

$$h = a^{-1}(X) = 1 - (X+s)^{-1}2s = (X+s)^{-1}(X-s)$$

satisfies $h^\dagger s h < s$ and lies in $\mathrm{H}(V_0^s)$. Thus $a : \mathrm{H}(V_0^s) \to \zeta_s^+(V_0)$ is a bijection. □

Corollary 5.3. — *The manifold* $\mathrm{H}(V_0^s)$ *is connected, and is closed under* $h \mapsto sh^\dagger s$.

Proof. — The space of all $X \in \mathrm{End}(V_0)$ with $\mathfrak{Re}\, X > 0$ is convex and hence connected. The connected property cannot be lost by removing the solution set of $\mathrm{Det}(X+s) = 0$; thus the space $\zeta_s^+(V_0)$ is connected. Clearly, the bijection $a^{-1} : \zeta_s^+(V_0) \to \mathrm{H}(V_0^s)$ is a continuous map, and it follows that $\mathrm{H}(V_0^s)$ is connected.

By Lem. 5.2, if $h \in \mathrm{H}(V_0^s)$, there exists $X \in \zeta_s^+(V_0)$ so that $h = 1 - (X+s)^{-1}2s$. Take the adjoint and conjugate by s to get $sh^\dagger s = 1 - (X^\dagger + s)^{-1}2s$. Since $\zeta_s^+(V_0)$ is obviously closed under $X \mapsto X^\dagger$, so is $\mathrm{H}(V_0^s)$ under $h \mapsto sh^\dagger s$. □

Remark. — It follows that if $h \in \mathrm{GL}(V_0)$ and $h^\dagger s h < s$ then one also has $hsh^\dagger < s$. In terms of the automorphism $\sigma : \mathrm{GL}(V_0) \to \mathrm{GL}(V_0)$ of (4.6), one can say that $\mathrm{H}(V_0^s)$ is closed under $h \mapsto \sigma(h)^{-1}$. By setting $V_0^+ = \mathbb{C}^p$ and $V_0^- = \mathbb{C}^q$ one gets the properties of the semigroup $\mathrm{H}_{p,q}$ which were stated without proof in Sects. 4.1.1 and 4.4.

5.2. Representation of $\mathrm{H}(V_0^s)$ and $\mathrm{U}(V_0^s)$ on $\mathscr{A}_V$. — Our approach is inspired by Howe's construction [**14**] of the Shale-Weil representation of the metaplectic group via the oscillator semigroup: we first construct a representation of the semigroup $\mathrm{H}(V_0^s)$, and then pass to its closure to obtain a representation of $\mathrm{U}(V_0^s) \subset \overline{\mathrm{H}(V_0^s)}$.

To begin, if a vector $v \in V_0$ is written according to the orthogonal decomposition $V_0 = V_0^+ \oplus V_0^-$ as $v = v_+ + v_-$, we assign to it a linear operator T_v on $\mathscr{A}_V$ by

$$\mathsf{T}_v = \mathrm{e}^{\mathrm{i}\mu(v_+)+\mathrm{i}\delta(cv_+)}\,\mathrm{e}^{\mathrm{i}\delta(v_-)+\mathrm{i}\mu(cv_-)}\;. \tag{5.2}$$

Since $\mathscr{A}_V$ is equipped with its canonical Hermitian structure – cf. Sect. 2.2 and (2.14) – in which the relations $\mu(v_+)^\dagger = \delta(cv_+)$ and $\delta(v_-)^\dagger = \mu(cv_-)$ hold, such operators are unitary: ${\mathsf{T}_v}^\dagger = \mathsf{T}_{-v} = {\mathsf{T}_v}^{-1}$. By a straightforward computation using the canonical commutation relations (2.12), one verifies the composition law

$$\mathsf{T}_u \mathsf{T}_v = \mathsf{T}_{u+v}\,\mathrm{e}^{-\mathrm{i}\mathfrak{Im}\langle u, sv\rangle}\;, \tag{5.3}$$

where $\langle\, ,\rangle$ is the Hermitian scalar product of the Hermitian vector space V_0. (The T_v with this law define a representation of the so-called Heisenberg group).

Now, with a rapidly decreasing function F on V_0 associate an operator $\mathrm{Op}(F)$ by

$$\mathrm{Op}(F) = \int_{V_0} F(v)\mathsf{T}_v\, d\mathrm{vol}(v) \tag{5.4}$$

where $d\mathrm{vol}(v)$ is Lebesgue measure on V_0 normalized by $\int_{V_0} \mathrm{e}^{-\frac{1}{2}\langle v,v\rangle} d\mathrm{vol}(v) = 1$. The multiplication law for such operators is readily seen to be

$$\mathrm{Op}(F_1)\mathrm{Op}(F_2) = \mathrm{Op}(F_1 \sharp F_2) \tag{5.5}$$

where $F_1\sharp F_2$ is the convolution product twisted by the multiplier in (5.3):

$$(F_1\sharp F_2)(u) = \int_{V_0} F_1(u-v)F_2(v)\,\mathrm{e}^{-\mathrm{i}\mathfrak{Im}\langle u,sv\rangle}\,d\mathrm{vol}(v)\,. \tag{5.6}$$

The next step is to specialize this convolution product to a good space of Gaussian functions on V_0. Recall the definition of the set $\zeta_s^+(V_0)\subset\mathrm{End}(V_0)$ from Lem. 5.2, and with every element $X\in\zeta_s^+(V_0)$ associate a Gaussian function $\gamma_X : V_0\to\mathbb{C}$ by

$$\gamma_X(v) = \mathrm{Det}(X+s)\,\mathrm{e}^{-\frac{1}{2}\langle v,Xv\rangle}\,. \tag{5.7}$$

If X and Y lie in $\zeta_s^+(V_0)$, then $\mathfrak{Re}(X+Y)>0$, so that the density

$$v\mapsto|\gamma_X(u-v)\gamma_Y(v)|\,d\mathrm{vol}(v)$$

decreases rapidly at infinity and the twisted convolution product $\gamma_X\sharp\gamma_Y$ exists.

Doing the Gaussian convolution integral by completing the square, we easily find

$$\gamma_X\sharp\gamma_Y = \gamma_{X\circ Y}\,, \tag{5.8}$$

with the composition law $(X,Y)\mapsto X\circ Y$ given by

$$X\circ Y := X-(X-s)(X+Y)^{-1}(X+s) = (Y+s)(X+Y)^{-1}(X+s)-s\,. \tag{5.9}$$

From Lem. 5.2 recall now the bijective map (we again abbreviate $\mathrm{Id}_{V_0}\equiv 1$)

$$a:\ \mathrm{H}(V_0^s)\to\zeta_s^+(V_0)\,,\quad x\mapsto s\,\frac{1+x}{1-x} = a_x\,.$$

Lemma 5.4. — *The mapping* $\mathsf{R}:\ \mathrm{H}(V_0^s)\to\mathrm{End}(\mathscr{A}_V)$ *defined by*

$$x\mapsto\mathsf{R}(x) = \mathrm{Det}(a_x+s)\int_{V_0}\mathrm{e}^{-\frac{1}{2}\langle v,a_xv\rangle}\,\mathsf{T}_v\,d\mathrm{vol}(v)$$

is a representation of the semigroup $\mathrm{H}(V_0^s)$ *by contractions, i.e.,* $\|\,\mathsf{R}(x)\,\|_{\mathrm{op}}\le 1$.

Proof. — With respect to multiplication in $\mathrm{H}(V_0^s)$ the function $x\mapsto a_x$ behaves as

$$\begin{aligned} a_{xy}+s &= 2s(1-xy)^{-1} = 2s\big((1-x)+(1-y)-(1-x)(1-y)\big)^{-1} \\ &= \big((a_x+s)^{-1}+(a_y+s)^{-1}-(a_x+s)^{-1}2s(a_y+s)^{-1}\big)^{-1} \\ &= (a_y+s)\,(a_x+a_y)^{-1}(a_x+s) = a_x\circ a_y+s\,, \end{aligned}$$

which is exactly the composition law (5.9). Thus the bijection $a^{-1}:\ \zeta_s^+(V_0)\to\mathrm{H}(V_0^s)$ transforms the complicated product (5.9) into plain composition of linear operators:

$$a^{-1}(X\circ Y) = a^{-1}(X)\,a^{-1}(Y)\,.$$

Setting $\mathsf{R}(x) := \mathrm{Op}(\gamma_{a_x})$ and using (5.8, 5.5) one gets

$$\mathsf{R}(xy) = \mathrm{Op}(\gamma_{a_x\circ a_y}) = \mathrm{Op}(\gamma_{a_x}\sharp\gamma_{a_y}) = \mathrm{Op}(\gamma_{a_x})\mathrm{Op}(\gamma_{a_y}) = \mathsf{R}(x)\mathsf{R}(y)\,,$$

which says that $x\mapsto\mathsf{R}(x)$ is a semigroup representation.

For the proof of the contraction property we refer to [**14**], Chapter 15. □

Now recall that the group of pseudo-unitary transformations $\mathrm{U}(V_0^s)$ defined in (4.3) is found in the closure $\overline{\mathrm{H}(V_0^s)}$. Hence we can define $\mathsf{R}'(g)$ for $g \in \mathrm{U}(V_0^s)$ by taking the limit of $\mathsf{R}(x_j)$ for any sequence $\{x_j\}_{j\in\mathbb{N}}$ in $\mathrm{H}(V_0^s)$ that converges to g. (Convergence here means convergence in a strong sense, namely w.r.t. the bounded strong* topology. For the details of this argument we refer to [**14**], Chapter 16.)

In particular, if a sequence $\{x_j\}_{j\in\mathbb{N}}$ in $\mathrm{H}(V_0^s)$ converges to an element g of the open and dense subset of $\mathrm{U}(V_0^s)$ where $1-g \in \mathrm{End}(V_0)$ is regular, then using $\mathsf{T}_v^\dagger = \mathsf{T}_{-v}$ and

$$a_x^\dagger = (1-x^\dagger)^{-1}2s - s = 2s(1-sx^\dagger s)^{-1} - s = a_{sx^\dagger s} ,$$

one sees that the limit operator $\mathsf{R}'(g) := \lim_{j\to\infty}\mathsf{R}(x_j)$ has an adjoint which is

$$\mathsf{R}'(g)^\dagger = \lim_{j\to\infty}\mathsf{R}(sx_j^\dagger s) = \mathsf{R}'(g^{-1}) ,$$

since $sg^\dagger s = g^{-1}$ for $g \in \mathrm{U}(V_0^s)$. Moreover, if $\{x_j\}$ and $\{y_j\}$ are two sequences in $\mathrm{H}(V_0^s)$ approaching g resp. h in $\mathrm{U}(V_0^s)$, then the semigroup law $\mathsf{R}(x_j)\mathsf{R}(y_j) = \mathsf{R}(x_j y_j)$ delivers the group law $\mathsf{R}'(g)\mathsf{R}'(h) = \mathsf{R}'(gh)$ by continuity of the limit.

We thus arrive at the following statement.

Lemma 5.5. — *Taking the limit (where still $\varepsilon\cdot 1 \equiv \varepsilon\,\mathrm{Id}_{V_0}$)*

$$\mathsf{R}'(g) = \lim_{\varepsilon\to 0+}\mathrm{Det}(a_g + s + \varepsilon\cdot 1)\int_{V_0} \mathrm{e}^{-\frac{1}{2}\langle v,(a_g+\varepsilon\cdot 1)v\rangle}\mathsf{T}_v\, d\mathrm{vol}(v)$$

yields a unitary representation $\mathsf{R}' : \mathrm{U}(V_0^s)\to\mathrm{U}(\mathscr{A}_V)$. This representation is compatible with the semigroup representation $\mathsf{R} : \mathrm{H}(V_0^s)\to\mathrm{GL}(\mathscr{A}_V)$ in the sense that $\mathsf{R}(xg) = \mathsf{R}(x)\mathsf{R}'(g)$ and $\mathsf{R}(gx) = \mathsf{R}'(g)\mathsf{R}(x)$ for all $x\in\mathrm{H}(V_0^s)$ and $g\in\mathrm{U}(V_0^s)$.

Since the construction of R' makes no reference to the representation of the Lie algebra $\mathfrak{u}(V_0^s)$ by the restriction $R_*|_{\mathfrak{u}(V_0^s)}$, their relationship is as yet an open question.

To answer it, pick any $1-X\equiv x\in\mathrm{H}(V_0^s)$ subject to the condition $X^\dagger s + sX > 0$. For such $X = 1-x$, the curve $t\mapsto x_t := 1-tX$ for $0<t\leq 1$ is a curve in $\mathrm{H}(V_0^s)$. Indeed, the condition for x_t to lie in $\mathrm{H}(V_0^s)$ is $x_t^\dagger s x_t < s$, which translates to

$$t^2 X^\dagger s X < t(X^\dagger s + sX) ,$$

and since $0 < X^\dagger s + sX$ and this inequality holds for $t=1$, it holds for all $0<t\leq 1$.

Note that X^{-1} exists, and that $\mathfrak{Re}\, Xs > 0$. Now insert $a_{x_t} = 2s(1-x_t)^{-1} - s = 2s(tX)^{-1} - s$ into the expression for $\mathsf{R}(x_t)$ from Lem. 5.4, and rescale $v\to\sqrt{t}\,v$:

$$\begin{aligned}\mathsf{R}(1-tX) \;=\; & \mathrm{Det}(tXs/2)^{-1}\int_{V_0}\mathrm{e}^{-\langle v,(Xs)^{-1}v\rangle}\mathrm{e}^{\mathrm{i}\sqrt{t}\,\mu(v_+)+\mathrm{i}\sqrt{t}\,\delta(cv_+)} \\ & \times \mathrm{e}^{\mathrm{i}\sqrt{t}\,\delta(v_-)+\mathrm{i}\sqrt{t}\,\mu(cv_-)}\,\mathrm{e}^{\frac{t}{2}\langle v,sv\rangle}\, d\mathrm{vol}(\sqrt{t}v) .\end{aligned}$$

Using $\mathrm{Det}(tXs/2)^{-1}d\mathrm{vol}(\sqrt{t}v) = \mathrm{Det}(Xs/2)^{-1}d\mathrm{vol}(v)$, take the one-sided derivative at $t = 0+$:

$$(d\mathsf{R})_e(-X) \;=\; -\mathrm{Det}(Xs/2)^{-1}\int_{V_0} \mathrm{e}^{-\langle v,(Xs)^{-1}v\rangle}\big(\mu(v_+)+\delta(v_-)\big) \\ \times\big(\delta(cv_+)+\mu(cv_-)\big)\,d\mathrm{vol}(v)\;.$$

To do the Gaussian integral over v, expand the integration vector v in orthonormal bases $\{e_i^+\}$ and $\{e_j^-\}$ of V_0^+ resp. V_0^-, and use $ce_i^\pm = \langle e_i^\pm,\cdot\rangle = f_i^\pm$. If X is decomposed into blocks according to $V_0 = V_0^+ \oplus V_0^-$, the final result assumes the form

$$(d\mathsf{R})_e\begin{pmatrix}\mathsf{A} & \mathsf{B}\\ \mathsf{C} & \mathsf{D}\end{pmatrix} = \sum\big(\mu(\mathsf{A}e_i^+)+\delta(\mathsf{C}e_i^+)\big)\delta(f_i^+) - \sum\big(\mu(\mathsf{B}e_j^-)+\delta(\mathsf{D}e_j^-)\big)\mu(f_j^-)\;.$$

This is exactly the operator which is assigned to $X = \sum(Xe_i^+)\otimes f_i^+ + \sum(Xe_j^-)\otimes f_j^-$ by the spinor-oscillator representation of Def. 2.7.

Since $\mathsf{R}' : \mathrm{U}(V_0^s) \to \mathrm{U}(\mathscr{A}_V)$ is constructed from $\mathsf{R} : \mathrm{H}(V_0^s) \to \mathrm{GL}(\mathscr{A}_V)$ by a continuous limit procedure, the same formula holds for $X \in \mathfrak{u}(V_0^s)$. We thus have:

Lemma 5.6. — *The infinitesimal of the representation* $\mathsf{R}' : \mathrm{U}(V_0^s) \to \mathrm{U}(\mathscr{A}_V)$ *agrees with the Lie superalgebra representation* $R_* : \mathfrak{gl}(V) \to \mathfrak{gl}(\mathscr{A}_V)$ *restricted to* $\mathfrak{u}(V_0^s)$:

$$(d\mathsf{R}')_e = R_*|_{\mathfrak{u}(V_0^s)}\;.$$

Another question to be addressed concerns the relation between the semigroup representation R and the Weyl action $\mathbf{w} : W_0 \to \mathrm{End}(\mathscr{A}_V)$ of Def. 2.7. To answer it in a good way, let us introduce for every $v+\varphi \in W_0$ the operator

$$\widetilde{\mathsf{T}}_{v+\varphi} = \mathrm{e}^{\mathrm{i}\mu(v_+)+\mathrm{i}\delta(v_-)}\,\mathrm{e}^{\mathrm{i}\delta(\varphi_+)-\mathrm{i}\mu(\varphi_-)}\;, \tag{5.10}$$

where $w = v+\varphi = v_+ + v_- + \varphi_+ + \varphi_-$ is the orthogonal decomposition according to $W_0 = V_0 \oplus V_0^* = V_0^+ \oplus V_0^- \oplus V_0^{+*} \oplus V_0^{-*}$. For $x \in \mathrm{GL}(V_0)$ recall $x\cdot w = xv + \varphi x^{-1}$.

Lemma 5.7. — *For all* $w \in W_0$ *and* $x \in \mathrm{H}(V_0^s) \subset \mathrm{GL}(V_0)$ *the following relation holds:*

$$\mathsf{R}(x)\widetilde{\mathsf{T}}_w = \widetilde{\mathsf{T}}_{xw}\mathsf{R}(x)\;.$$

Remark. — Being unbounded, $\widetilde{\mathsf{T}}_{v+\varphi}$ is not an operator on the L^2-space of $\mathscr{A}_V$. Still, it does exist as an operator on the algebra of holomorphic functions on $V_0^+ \oplus (V_0^-)^*$.

Proof. — By the canonical commutation relations (2.12), the operators $\widetilde{\mathsf{T}}_{v+\varphi}$ satisfy

$$\widetilde{\mathsf{T}}_{v+\varphi}\widetilde{\mathsf{T}}_{v'+\varphi'} = \widetilde{\mathsf{T}}_{v+v'+\varphi+\varphi'}\,\mathrm{e}^{-\varphi(v')}\;.$$

Using this composition law and the identity $\mathsf{T}_u = \mathrm{e}^{-\frac{1}{2}\langle u,su\rangle}\widetilde{\mathsf{T}}_{u+csu}$ in the defining expression for $\mathsf{R}(x)$ in Lem. 5.4, one writes the product $\mathsf{R}(x)\widetilde{\mathsf{T}}_{v+\varphi}$ in the form

$$\mathsf{R}(x)\widetilde{\mathsf{T}}_{v+\varphi} = \mathrm{Det}(a_x+s)\int_{V_0}\mathrm{e}^{-\frac{1}{2}\langle u,(a_x+s)u\rangle+\varphi((1-x)^{-1}u)}\widetilde{\mathsf{T}}_{u+xv+csu}\,d\mathrm{vol}(u)\;.$$

Here a shift of integration variables $u \to u-(1-x)v$ and $csu \to csu-\varphi$ has been made, and the relation $\frac{1}{2}(a_x+s)(1-x) = s$ was used. Similarly, one evaluates the product

$\widetilde{\mathsf{T}}_{xv+\varphi x^{-1}}\mathsf{R}(x)$ by shifting $csu \to csu - \varphi x^{-1}$. The resulting expression is the same as for $\mathsf{R}(x)\widetilde{\mathsf{T}}_{v+\varphi}$. Since $xv + \varphi x^{-1} = x\cdot(v+\varphi)$, this already proves the lemma. □

Corollary 5.8. — *The semigroup representation* R *is compatible with the Weyl action* $\mathbf{w} : W_0 \to \mathrm{End}(\mathscr{A}_V)$ *of Def. 2.7 in the sense that*

$$\mathsf{R}(x)\mathbf{w}(w) = \mathbf{w}(xw)\mathsf{R}(x)$$

for all $w \in W_0$ *and* $x \in \mathrm{H}(V_0^s)$.

Proof. — Differentiating the curve of operators $t \mapsto \widetilde{\mathsf{T}}_{tw}$ at $t=0$ gives the Weyl action:

$$\frac{d}{dt}\Big|_{t=0}\widetilde{\mathsf{T}}_{tw} = \mathrm{i}\mu(v_+) + \mathrm{i}\delta(v_-) + \mathrm{i}\delta(\varphi_+) - \mathrm{i}\mu(\varphi_-) = \mathrm{i}\mathbf{w}(w)\ .$$

Therefore the assertion follows by differentiating the statement of Lem. 5.7. □

One final statement is needed to clarify the relation of R with earlier structure.

Lemma 5.9. — *For every* $x \in \mathrm{H}(V_0^s)$ *and* $Y \in \mathfrak{gl}(V_0)$ *there exists some* $\varepsilon > 0$ *so that* $\mathrm{e}^{tY}x \in \mathrm{H}(V_0^s)$ *for all* $t \in [-\varepsilon, \varepsilon]$. *The derivative of* R *along the curve* $t \mapsto \mathrm{e}^{tY}x$ *in* x *is*

$$\frac{d}{dt}\mathsf{R}(\mathrm{e}^{tY}x)\Big|_{t=0} = R_*(Y)\mathsf{R}(x)\ ,$$

where $R_* \equiv R_*|_{\mathfrak{gl}(V_0)}$ *is the restriction of the representation* $R_* : \mathfrak{gl}(V) \to \mathfrak{gl}(\mathscr{A}_V)$.

Proof. — The first statement just says that the semigroup $\mathrm{H}(V_0^s)$ is open in $\mathrm{GL}(V_0)$.

For the second statement use the formula

$$\frac{d}{dt}a_{\mathrm{e}^{tY}x}\Big|_{t=0} = 2s\frac{d}{dt}(1-\mathrm{e}^{tY}x)^{-1}\Big|_{t=0} = (a_x+s)Yx(1-x)^{-1}$$

to compute the derivative as

$$\begin{aligned}\frac{d}{dt}\mathsf{R}(\mathrm{e}^{tY}x)\Big|_{t=0} &= \mathrm{Det}(a_x+s)\int_{V_0}\mathrm{e}^{-\frac{1}{2}\langle u,a_xu\rangle}f_{Y,x}(u)\mathsf{T}_u\,d\mathrm{vol}(u)\ ,\\ f_{Y,x}(u) &= \mathrm{Tr}_{V_0}Yx(1-x)^{-1} - \tfrac{1}{2}\big\langle u,(a_x+s)Yx(1-x)^{-1}u\big\rangle\ .\end{aligned}$$

On the other hand, identifying $\mathfrak{gl}(V_0) \simeq V_0 \otimes V_0^*$ let $Y = v\otimes\varphi$ and notice

$$R_*(v\otimes\varphi) = \mathbf{w}(v)\mathbf{w}(\varphi) = -\frac{\partial^2}{\partial t\partial t'}\widetilde{\mathsf{T}}_{tv+t'\varphi}\Big|_{t=t'=0}\ .$$

In order to exploit this relation, consider the product

$$\widetilde{\mathsf{T}}_{v+\varphi}\mathsf{R}(x) = \mathrm{Det}(a_x+s)\int_{V_0}\mathrm{e}^{-\frac{1}{2}\langle u,(a_x+s)u\rangle-\varphi(u)}\,\widetilde{\mathsf{T}}_{u+v+csu+\varphi}\,d\mathrm{vol}(u)\ .$$

Now make a shift of integration variables $u \to u - v$ and $csu \to csu - \varphi$, and use the identity $\frac{s}{2}(a_x+s) - 1 = x(1-x)^{-1}$ to obtain

$$\widetilde{\mathsf{T}}_{v+\varphi}\mathsf{R}(x) = \mathrm{Det}(a_x+s)\int_{V_0}\mathrm{e}^{-\frac{1}{2}\langle u,a_xu\rangle+\frac{1}{2}\langle u,(a_x+s)v\rangle+\varphi(x(1-x)^{-1}(u-v))}\mathsf{T}_u\,d\mathrm{vol}(u)\ .$$

Substitute $v \to tv$ and $\varphi \to t'\varphi$ and differentiate at $t = t' = 0$ to arrive at a formula for $R_*(v\otimes\varphi)\mathsf{R}(x)$. The result is the same as our expression for the derivative of $\mathsf{R}(\mathrm{e}^{tY}x)$

at $t=0$ specialized to the case $Y = v \otimes \varphi$. The formula of the lemma now follows because every $Y \in \mathfrak{gl}(V_0)$ can be expressed as a linear combination $Y = \sum v_i \otimes \varphi_i$. $\square$

Lemma 5.10. — *Let P be any element of the Clifford-Weyl algebra of $W = V \oplus V^*$ acting on $\mathscr{A}_V$ by the Clifford-Weyl action $\mathbf{q}$ of Def. 2.7. Then the operator $\mathbf{q}(P)\mathsf{R}(x)$ for $x \in \mathrm{H}(V_0^s)$ is trace-class and $\mathrm{STr}_{\mathscr{A}_V}\,\mathbf{q}(P)\mathsf{R}(x)$ depends analytically on x.*

Proof. — The representation space $\mathscr{A}_V$ is isomorphic to the tensor product of $\mathscr{A}_{V_1} := \wedge(V_1^+ \oplus V_1^{-*})$ with $\mathscr{A}_{V_0} := \mathsf{S}(V_0^+ \oplus V_0^{-*})$, and the operator $\mathsf{R}(x)$ acts non-trivially only on the second factor. Since the first factor $\mathscr{A}_{V_1}$ is a finite-dimensional vector space, the assertion is true if $\mathrm{Tr}_{\mathscr{A}_{V_0}}\mathbf{w}(P)\mathsf{R}(x)$ exists and depends analytically on x for every Weyl algebra element $P \in \mathfrak{w}(W_0)$, operating by the Weyl action $\mathbf{w}$ on $\mathscr{A}_{V_0}$.

Applying the unitary operator T_v to $\phi_0 \equiv 1 \in \mathbb{C} \subset \mathscr{A}_{V_0}$ one gets a unit vector $\phi_v := \mathsf{T}_v \phi_0 \in \mathscr{A}_{V_0}$, which is called a *coherent state*. From the relation (5.3) one has

$$(\phi_v\,, \mathsf{T}_u\,\phi_v) = (\phi_0\,, \mathsf{T}_{-v}\mathsf{T}_u\mathsf{T}_v\phi_0) = \mathrm{e}^{-2\mathrm{i}\mathfrak{Im}\langle u,sv\rangle}(\phi_0\,, \mathsf{T}_u\phi_0) = \mathrm{e}^{-2\mathrm{i}\mathfrak{Im}\langle u,sv\rangle - \frac{1}{2}\langle u,u\rangle}\ ,$$

where $(\,,\,)$ denotes the Hermitian scalar product of $\mathscr{A}_{V_0}$.

We will show that $\mathrm{Tr}_{\mathscr{A}_{V_0}}\mathbf{w}(P)\mathsf{R}(x) < \infty$ by integrating over coherent states. First, consider $(\phi_v\,, \mathsf{R}(x)\phi_v)$, using for $\mathsf{R}(x)$ the formula of Lem. 5.4. Since $|\mathrm{e}^{-2\mathrm{i}\mathfrak{Im}\langle u,sv\rangle}| = 1$ and the Gaussian density $u \mapsto \mathrm{e}^{-\frac{1}{2}\langle u,\mathfrak{Re}(a_x)u\rangle}\mathrm{e}^{-\frac{1}{2}\langle u,u\rangle}d\mathrm{vol}(u)$ has finite integral over V_0, one may take the coherent-state expectation inside the integral to obtain

$$(\phi_v\,, \mathsf{R}(x)\phi_v) = \mathrm{Det}(a_x + s)\int_{V_0} \mathrm{e}^{-\frac{1}{2}\langle u,(a_x+1)u\rangle - 2\mathrm{i}\mathfrak{Im}\langle u,sv\rangle}\ d\mathrm{vol}(u)\ .$$

The Gaussian integral over $u \in V_0$ converges absolutely, and performing it by completing the square one gets

$$(\phi_v\,, \mathsf{R}(x)\phi_v) = \mathrm{Det}(a_x + s)\,\mathrm{Det}(a_x + 1)^{-1}\,\mathrm{e}^{-\frac{1}{2}\langle sv,\,(a_x+1)^{-1}sv\rangle}\ ,$$

which is a rapidly decreasing function of v.

Now the Heisenberg group action $\mathrm{U}_1 \times V_0 \to \mathrm{U}(\mathscr{A}_{V_0})$, $(z\,,u) \mapsto z\,\mathsf{T}_u$, is well known (as part of the Stone-von Neumann Theorem) to be irreducible on $\mathscr{A}_{V_0}$. Therefore, if $L \in \mathrm{End}(\mathscr{A}_{V_0})$ is a trace-class operator, its trace is computed by the integral

$$\mathrm{Tr}_{\mathscr{A}_{V_0}} L = \int_{V_0} (\phi_v\,, L\,\phi_v)\,d\mathrm{vol}(v) \Big/ \int_{V_0} \mathrm{e}^{-\langle v,v\rangle} d\mathrm{vol}(v)\ .$$

Indeed, the function $L \mapsto \int_{V_0}(\phi_v\,, L\,\phi_v)\,d\mathrm{vol}(v)$ does not change when L is conjugated by T_u, and by the irreducibility of $u \mapsto \mathsf{T}_u$ any such function must be proportional to the operation of taking the trace. The constant of proportionality is determined by inserting some rank-one projector, say $L = \phi_0(\phi_0\,,\cdot)$, and using $|(\phi_0\,,\phi_v)|^2 = \mathrm{e}^{-\langle v,v\rangle}$.

Since $v \mapsto (\phi_v\,, \mathsf{R}(x)\phi_v)$ is a decreasing Gaussian function and $x \mapsto a_x$ depends analytically on $x \in \mathrm{H}(V_0^s)$, it is now clear that the trace of $\mathsf{R}(x)$ exists and is analytic in x. An easy computation gives

$$\mathrm{Tr}_{\mathscr{A}_{V_0}}\mathsf{R}(x) = \mathrm{Det}\big(\tfrac{1}{2}(a_x + s)\big) = \mathrm{Det}(s)\,\mathrm{Det}(1 - x)^{-1}\ .$$

Essentially the same argument goes through for the case of $\mathbf{w}(P)\mathsf{R}(x)$ with a polynomial $P \in \mathfrak{w}(W_0)$. Indeed, since the expectation $(\phi_v, \mathbf{w}(P)\mathsf{T}_u \phi_v)$ is easily verified to be $\mathrm{e}^{-2\mathrm{i}\mathfrak{Im}\langle u, sv\rangle}$ times a polynomial in u and v of finite degree, $(\phi_v, \mathbf{w}(P)\mathsf{R}(x)\phi_v)$ still is a rapidly decreasing function of v and therefore its integral – the trace of $\mathbf{w}(P)\mathsf{R}(x)$ – exists and depends analytically on x. □

5.3. Combining the representations. — Let us piece together the various building stones we have accumulated. From Lem. 5.4 we have a representation R of the semigroup $\mathrm{H}(V_0^s)$ which we now rename to

$$R_0 : \mathrm{H}(V_0^s) \to \mathrm{End}(\mathscr{A}_V) .$$

Notice that R_0 coincides with the representation $\omega \otimes \tilde{\omega}$ of Sect. 2.3 if $\mathrm{H}(V_0^s)$ is restricted to $\mathrm{H}^<(V_0^+) \times \mathrm{H}^>(V_0^-) \hookrightarrow \mathrm{H}(V_0^s)$. We also recall that we have a $\mathrm{GL}(V_1)$-representation

$$R_1 : \mathrm{GL}(V_1) \to \mathrm{GL}(\mathscr{A}_V) ,$$

which agrees with $\sigma \otimes \tilde{\sigma}$ of Sect. 2.1 upon restriction to $\mathrm{GL}(V_1^+) \times \mathrm{GL}(V_1^-) \hookrightarrow \mathrm{GL}(V_1)$.

Combining R_1 with R_0 we have

$$R : \mathrm{GL}(V_1) \times \mathrm{H}(V_0^s) \to \mathrm{End}(\mathscr{A}_V) , \quad (g,h) \mapsto R_1(g)R_0(h) . \tag{5.11}$$

As a corollary to Lem. 5.5 we also have a unitary representation of the direct product of real Lie groups $\mathrm{U}(V_1) \times \mathrm{U}(V_0^s)$. We denote this by

$$R' : \mathrm{U}(V_1) \times \mathrm{U}(V_0^s) \to \mathrm{U}(\mathscr{A}_V) . \tag{5.12}$$

In the next statement, the adjoint action $\mathrm{Ad}(x) : \mathfrak{gl}(V) \to \mathfrak{gl}(V)$ for $V = V_1 \oplus V_0$ is meant to be conjugation by $x \in \mathrm{GL}(V_1) \times \mathrm{H}(V_0^s) \hookrightarrow \mathrm{GL}(V)$.

Proposition 5.11. — *The semigroup representation* $R : \mathrm{GL}(V_1) \times \mathrm{H}(V_0^s) \to \mathrm{End}(\mathscr{A}_V)$ *and the Lie superalgebra representation* $R_* : \mathfrak{gl}(V) \to \mathfrak{gl}(\mathscr{A}_V)$ *are compatible in that*

$$R(x)\,R_*(Y) = R_*(\mathrm{Ad}(x)Y)\,R(x)$$

for all $Y \in \mathfrak{gl}(V)$ *and* $x \in \mathrm{GL}(V_1) \times \mathrm{H}(V_0^s)$.

Proof. — Let $\mathbf{c} + \mathbf{w} : W_1 \oplus W_0 \to \mathrm{End}(\mathscr{A}_V)$ be the Clifford-Weyl action of Def. 2.7. If $x = (x_1, x_0) \in \mathrm{GL}(V_1) \times \mathrm{H}(V_0^s)$, then by combining Lem. 5.1 and Cor. 5.8 we have

$$R(x)\big(\mathbf{c}(w_1) + \mathbf{w}(w_0)\big) = \big(\mathbf{c}(x_1 \cdot w_1) + \mathbf{w}(x_0 \cdot w_0)\big)R(x) . \tag{5.13}$$

By definition, the representation R_* is induced from the spinor-oscillator representation of the Clifford-Weyl algebra $\mathfrak{q}(W)$, which in turn is determined by the Clifford-Weyl action $\mathbf{c} + \mathbf{w}$. The statement therefore follows from the property (5.13) of $\mathbf{c} + \mathbf{w}$ in combination with the fact that by the identification $\mathfrak{gl}(V) \simeq V \otimes V^*$ the action of $\mathrm{GL}(V_1) \times \mathrm{GL}(V_0) \hookrightarrow \mathrm{GL}(V)$ on $W = V \oplus V^*$ by $x(v + \varphi) = xv + \varphi x^{-1}$ corresponds to the adjoint action $\mathrm{Ad}(x)(v \otimes \varphi) = x(v \otimes \varphi)x^{-1}$. □

Corollary 5.12. — *The group representation* $R' : \mathrm{U}(V_1) \times \mathrm{U}(V_0^s) \to \mathrm{U}(\mathscr{A}_V)$ *and the superalgebra representation* $R_* : \mathfrak{gl}(V) \to \mathfrak{gl}(\mathscr{A}_V)$ *are compatible in the sense that*

$$R'(g) R_*(X) R'(g)^{-1} = R_*(\mathrm{Ad}(g)X)$$

for all $X \in \mathfrak{gl}(V)$ *and* $g \in \mathrm{GL}(V_1) \times \mathrm{U}(V_0^s)$.

Proof. — $R'(g)$ for $g \in \mathrm{U}(V_1) \times \mathrm{U}(V_0^s)$ has an inverse. Thus the assertion follows from Prop. 5.11 by sending $x \to g$ and multiplying with $R'(g)^{-1}$ on the right. □

The next result is an easy consequence of Lem. 5.9 in the present, combined setting.

Proposition 5.13. — *If* $x \in \mathrm{GL}(V_1) \times \mathrm{H}(V_0^s)$ *and* $Y \in \mathfrak{g}_0 \equiv \mathfrak{gl}(V)_0$, *the directional derivative of* R *at* x *along* Y *exists and is given by*

$$\frac{d}{dt} R(\mathrm{e}^{tY} x)\Big|_{t=0} = R_*(Y) R(x) \; .$$

5.4. Proof of Prop. 4.4. — We finally tackle the true goal of the current section and return to the Howe pair setting based on the decomposition $V = U \otimes \mathbb{C}^N$ and its refinement $V_\tau^\pm = U_\tau^\pm \otimes \mathbb{C}^N$ with $U_0^+ = U_1^+ = \mathbb{C}^p$ and $U_0^- = U_1^- = \mathbb{C}^q$, and $p + q = n$.

From the tensor-product decompositions $V_1 = \mathbb{C}^n \otimes \mathbb{C}^N$ and $V_0 = \mathbb{C}^{p+q} \otimes \mathbb{C}^N$ we have the embeddings

$$\mathrm{GL}_n \times \mathrm{U}_N \hookrightarrow \mathrm{GL}(V_1) \; , \quad \mathrm{H}_{p,q} \times \mathrm{U}_N \hookrightarrow \mathrm{H}(V_0^s) \; ,$$

which by $V_1 \oplus V_0 = (U_1 \oplus U_0) \otimes \mathbb{C}^N$ combine to a semigroup embedding

$$(\mathrm{GL}_n \times \mathrm{H}_{p,q}) \times \mathrm{U}_N \hookrightarrow \mathrm{GL}(V_1) \times \mathrm{H}(V_0^s) \; .$$

Similarly, we have a Lie group embedding

$$(\mathrm{U}_n \times \mathrm{U}_{p,q}) \times \mathrm{U}_N \hookrightarrow \mathrm{U}(V_1) \times \mathrm{U}(V_0^s) \; .$$

Using these, we now restrict and project the $\mathfrak{gl}(V)$-representation $(\mathscr{A}_V, R_*, R, R')$ to a $\mathfrak{gl}(U)$-representation $(\mathscr{V}_\lambda, \rho_*, \rho, \rho')$ on the subalgebra of U_N-invariants $\mathscr{V}_\lambda \subset \mathscr{A}_V$. The semigroup representation ρ is defined by

$$\rho : \; \mathrm{GL}_n \times \mathrm{H}_{p,q} \to \mathrm{GL}(\mathscr{V}_\lambda) \; , \quad x \mapsto R(x, \mathrm{Id}_N)|_{\mathscr{V}_\lambda} \; ,$$

and the representation ρ' of the Lie group $G_\mathbb{R} = \mathrm{U}_n \times \mathrm{U}_{p,q}$ is defined in the same way:

$$\rho' : \; \mathrm{U}_n \times \mathrm{U}_{p,q} \to \mathrm{U}(\mathscr{V}_\lambda) \; , \quad g \mapsto R'(g, \mathrm{Id}_N)|_{\mathscr{V}_\lambda} \; .$$

This gives the existence statement of Prop. 4.4. The stated compatibility of ρ and ρ' with ρ_* is a consequence of the compatibility of R and R' with R_*.

To go further and prove the key formula (4.24), recall the setting of Sect. 4.1: we are given a real-analytic manifold $M = M_1 \times M_0 \subset \mathrm{GL}(U_1) \times \mathrm{GL}(U_0)$, a $G_\mathbb{R}$-principal bundle $P \to M$, an associated vector bundle $F \to M$ with standard fibre $\mathfrak{g}_{\mathbb{R},1} \equiv \mathfrak{u}(U^s)_1$, and a sheaf of graded-commutative algebras $\mathscr{F} = \Gamma(M, \mathbb{C} \otimes \wedge F^*)$. The next goal is to describe the character of the representation $(\mathscr{V}_\lambda, \rho_*, \rho, \rho')$ as a section of $\mathscr{F}$.

Our first observation is that, if $\{\mathsf{F}_i\}$ is a basis of $\mathfrak{g}_{\mathbb{R},1}$ and $\{\varphi^i\}$ is the corresponding basis of generators of $\wedge(\mathfrak{g}^*_{\mathbb{R},1})$, the operator

$$\rho(\Xi) \equiv \rho(x\,\mathrm{e}^{\xi} y^{-1}) := \rho(x)\,\mathrm{e}^{\sum \varphi^i \rho_*(\mathsf{F}_i)} \rho(y^{-1}) \;, \tag{5.14}$$

which is associated to the supermatrix $\Xi = x\,\mathrm{e}^{\xi} y^{-1}$ by the representation $(\mathscr{V}_\lambda, \rho_*, \rho, \rho')$, makes good sense. Indeed, if we replace $(x,y) \in P$ by (xg, yg) with $g \in G_{\mathbb{R}}$ and the generators φ^i of $\wedge(\mathfrak{g}^*_{\mathbb{R},1})$ by the transformed generators $\sum \varphi^j \mathrm{Ad}^*(g^{-1})_j^{\;i}$, then

$$\rho(xg)\mathrm{e}^{\sum \varphi^j \mathrm{Ad}^*(g^{-1})_j^{\;i} \rho_*(\mathsf{F}_i)} \rho((yg)^{-1}) = \rho(x)\,\mathrm{e}^{\sum \varphi^i \rho_*(\mathsf{F}_i)} \rho(y^{-1}) \;,$$

since $\rho(xg) = \rho(x)\rho'(g)$ and $\rho'(g)\rho_*(\mathsf{F}_i)\rho'(g)^{-1} = \rho_*(\mathrm{Ad}(g)\mathsf{F}_i) = \sum \rho_*(\mathsf{F}_j)\mathrm{Ad}(g)^j_{\;i}$.

As a result, the character $\mathrm{STr}_{\mathscr{V}_\lambda} \rho(\Xi)$ is a $G_{\mathbb{R}}$-equivariant mapping from P to the exterior algebra $\mathbb{C} \otimes \wedge(\mathfrak{g}^*_{\mathbb{R},1})$, and thus defines a section $\chi \in \mathscr{F}$ via the usual isomorphism:

$$\chi(xy^{-1}) = \big[(x,y);\, \mathrm{STr}_{\mathscr{V}_\lambda} \rho(x)\,\mathrm{e}^{\sum \varphi^i \rho_*(\mathsf{F}_i)} \rho(y^{-1})\big] \;. \tag{5.15}$$

By its definition as a character, this section χ is radial, i.e., one has $\chi(gmg^{-1}) = \chi(m)$ for all $g \in G_{\mathbb{R}}$, and $(\widehat{Y}^L + \widehat{Y}^R)\chi = 0$ for all $Y \in \mathfrak{g}_{\mathbb{R},1}$.

Lemma 5.14. — *The section $\chi \in \mathscr{F}$ defined by Eq. (5.15) is radial and analytic.*

Proof. — It remains to establish the analyticity of χ. For that purpose, fixing some pair $(x,y) \in P$ in the fibre over the point $xy^{-1} = m = (m_1, m_0) \in M$, use the compatibility statement of Prop. 5.11 to write

$$\mathrm{STr}_{\mathscr{V}_\lambda} \rho(\Xi) = \mathrm{STr}_{\mathscr{V}_\lambda} \rho(x)\,\mathrm{e}^{\sum \varphi^i \rho_*(\mathsf{F}_i)} \rho(y^{-1}) = \mathrm{STr}_{\mathscr{V}_\lambda} \mathrm{e}^{\sum \varphi^i \rho_*(\mathrm{Ad}(x)\mathsf{F}_i)} \rho(m) \;.$$

By the nilpotency of the generators φ^i, the power series expansion of the exponential function produces only a finite number of terms

$$\sum \varphi^{i_1} \cdots \varphi^{i_k}\, \mathrm{STr}_{\mathscr{V}_\lambda} \rho_*(\mathrm{Ad}(x)\mathsf{F}_{i_1}) \cdots \rho_*(\mathrm{Ad}(x)\mathsf{F}_{i_k})\, \rho(m) \;.$$

The numerical coefficients of these terms are of the form $\mathrm{STr}_{\mathscr{V}_\lambda} \mathbf{q}(P)\rho(m)$ where P is a polynomial in the Clifford-Weyl algebra of $W = V \oplus V^*$. By straightforward adaptation of Lem. 5.10 to the present situation, they depend analytically on $m \in M$ through $\rho(m)$. They also depend analytically on the factor x of $m = xy^{-1}$ through $\mathrm{Ad}(x)$. ☐

The next statement is the main step of completing the proof of Prop. 4.4, as it will allow us to lift the equality of functions of Cor. 4.1 to an equality of sections of $\mathscr{F}$.

Proposition 5.15. — *An analytic radial section $\vartheta \in \mathscr{F}$ is already determined by its numerical part,* $\mathrm{num}(\vartheta) \in \Gamma(M, \mathbb{C})$.

Proof. — Again, for $xy^{-1} = m = (m_1, m_0) \in M$ write

$$\Xi = x\,\mathrm{e}^{\sum \varphi^i \otimes \mathsf{F}_i} y^{-1} = \mathrm{e}^{\sum \varphi^i \otimes \mathrm{Ad}(x)\mathsf{F}_i}\, m \;.$$

Note that $\mathrm{Ad}(x)$ in general sends $\mathsf{F}_i \in \mathfrak{g}_{\mathbb{R},1}$ into the complex space $\mathfrak{g}_1 = \mathfrak{g}_{\mathbb{R},1} + \mathrm{i}\mathfrak{g}_{\mathbb{R},1}$.

From Lem. 4.8 we know that the spectrum of $m_0 \in M_0$ lies on the positive real axis in $\mathbb{C}$ while avoiding unity; whereas the spectrum of $m_1 \in M_1$ lies on the unit circle.

Thus the spectral sets of m_0 and m_1 are disjoint. As a consequence, m is regular in the sense of Prop. 3.8 and the supermatrix $\mathrm{e}^{\sum \varphi^i \otimes \mathrm{Ad}(x)\mathsf{F}_i} m$ can be block diagonalized.

This means that if $\mathfrak{g}_0 = \mathfrak{gl}(U)_0$ and $\mathfrak{g}_1 = \mathfrak{gl}(U)_1$, there exist complex Grassmann envelope elements $\Xi_1 \in \wedge^{\mathrm{odd}}(\mathfrak{g}_1^*) \otimes \mathfrak{g}_1$ and $\Xi_2 \in \wedge^{\mathrm{even}}(\mathfrak{g}_1^*) \otimes \mathfrak{g}_0$ such that

$$\mathrm{e}^{\sum \varphi^i \otimes \mathrm{Ad}(x)\mathsf{F}_i} m = \mathrm{e}^{\Xi_1}\, \mathrm{e}^{\Xi_2} m\, \mathrm{e}^{-\Xi_1} \; .$$

From the proof of Prop. 3.8 we also know how to construct these nilpotent elements: we pass to the equivalent equation

$$\sum \varphi^i \otimes \mathrm{Ad}(x)\mathsf{F}_i = \ln \left(\mathrm{e}^{\Xi_1} \mathrm{e}^{\Xi_2} \mathrm{e}^{-\mathrm{Ad}(m)\Xi_1} \right) ,$$

and, after expanding the right-hand side by the Baker-Campbell-Hausdorff formula, solve this equation for Ξ_1 and Ξ_2 in terms of $\sum \varphi^i \otimes \mathrm{Ad}(x)\mathsf{F}_i$. Given the basis $\{\mathsf{F}_i\}$ of $\mathfrak{g}_{\mathbb{R},1}$ and fixing some basis $\{\mathsf{E}_a\}$ of $\mathfrak{g}_{\mathbb{R},0}$, we write

$$\Xi_1 = \sum \xi_1^i \otimes \mathsf{F}_i \; , \quad \Xi_2 = \sum \xi_2^a \otimes \mathsf{E}_a \; ,$$

where ξ_2^a are even sections of $\mathscr{F}$, and the functions ξ_1^i are odd.

These preparations do not yet depend on the analytic and radial property of $\vartheta \in \mathscr{F}$. The rest of the argument, for clarity, will first be given for the case of $\vartheta = \chi$ of (5.15), which is one of the two cases of relevance below.

The algorithm for constructing Ξ_1 and Ξ_2 uses the bracket of $\mathfrak{gl}(U)$, and nothing but that bracket. Therefore, exactly the same procedure goes through at the representation level, and we have

$$\sum \varphi^i \rho_*(\mathrm{Ad}(x)\mathsf{F}_i) = \ln \left(\mathrm{e}^{\sum \xi_1^i \rho_*(\mathsf{F}_i)} \mathrm{e}^{\sum \xi_2^a \rho_*(\mathsf{E}_a)} \mathrm{e}^{-\sum \xi_1^i \rho_*(\mathrm{Ad}(m)\mathsf{F}_i)} \right) .$$

Exponentiate both sides, multiply by $\rho(m)$ on the right, and move $\rho(m)$ past the right-most exponential by Prop. 5.11. Then take the supertrace,

$$\mathrm{STr}_{\mathscr{V}_\lambda} \mathrm{e}^{\sum \varphi^i \rho_*(\mathrm{Ad}(x)\mathsf{F}_i)} \rho(m) = \mathrm{STr}_{\mathscr{V}_\lambda} \mathrm{e}^{\sum \xi_1^i \rho_*(\mathsf{F}_i)} \mathrm{e}^{\sum \xi_2^a \rho_*(\mathsf{E}_a)} \rho(m)\, \mathrm{e}^{-\sum \xi_1^i \rho_*(\mathsf{F}_i)} ,$$

and use the cyclic property of STr (or, equivalently, the radial property of χ) to obtain

$$\mathrm{STr}_{\mathscr{V}_\lambda} \rho(\Xi) = \mathrm{STr}_{\mathscr{V}_\lambda} \mathrm{e}^{\sum \varphi^i \rho_*(\mathrm{Ad}(x)\mathsf{F}_i)} \rho(m) = \mathrm{STr}_{\mathscr{V}_\lambda} \mathrm{e}^{\sum \xi_2^a \rho_*(\mathsf{E}_a)} \rho(m) \; .$$

By Prop. 5.13 the last expression can be expressed as a finite sum of derivatives via the $\mathfrak{g}_0$-action on the left, $\mathsf{E}_a \mapsto \widehat{\mathsf{E}}_a^L$:

$$\mathrm{STr}_{\mathscr{V}_\lambda} \mathrm{e}^{\sum \xi_2^a \rho_*(\mathsf{E}_a)} \rho(m) = \mathrm{e}^{-\sum \xi_2^a \widehat{\mathsf{E}}_a^L}\, \mathrm{STr}_{\mathscr{V}_\lambda} \rho(m) \; .$$

Since the function $m \mapsto \mathrm{STr}_{\mathscr{V}_\lambda} \rho(m)$ is the numerical part $\mathrm{num}(\chi)$ of (5.15), we have proved for this case that

$$\chi = \mathrm{e}^{-\sum \xi_2^a \widehat{\mathsf{E}}_a^L}\, \mathrm{num}(\chi) \; ,$$

i.e., given the even sections ξ_2^a and derivations $\widehat{\mathsf{E}}_a^L$ (where of course the sum $\sum \xi_2^a \widehat{\mathsf{E}}_a^L$ does not depend on the choice of basis but is invariantly defined), χ is determined by its numerical values and a finite number of derivatives thereof.

In order for this argument to go through, all that is needed is radiality of χ, to cancel the first factor against the last one in $\mathrm{e}^{\Xi_1}\mathrm{e}^{\Xi_2} m\, \mathrm{e}^{-\Xi_1}$, and analyticity of χ, to convert Ξ_2

in $e^{\Xi_2} m$ into a differential operator. It is therefore clear that the same reasoning applies to any section $\vartheta \in \mathscr{F}$ which is radial and analytic. □

Finally, recall from Def. 4.3 exactly what is meant by the reciprocal of the superdeterminant of $\mathrm{Id}_V - \Xi \otimes u$, and let $\vartheta \in \mathscr{F}$ be the section which is given by the integral on the right-hand side of (4.24):

$$\vartheta(xy^{-1}) := \left[(x,y); \int_{\mathrm{U}_N} \mathrm{SDet}\big(\mathrm{Id}_V - x\,\mathrm{e}^{\sum \varphi^i \otimes \mathsf{F}_i} y^{-1} \otimes u\big)^{-1} du\right].$$

This section ϑ is radial, because the superdeterminant has the property of being radial; and since $\mathrm{SDet}(\mathrm{Id}_V - \Xi \otimes u)^{-1}$ is analytic in the supermatrix $\Xi = x\,\mathrm{e}^{\sum \varphi^i \otimes \mathsf{F}_i} y^{-1}$ for every fixed unitary matrix u, so is ϑ, the result of integrating over $u \in \mathrm{U}_N$.

It now follows that ϑ equals the character χ of (5.15). Indeed, both ϑ and χ are radial and analytic, and their numerical parts agree by Cor. 4.1. Therefore, by Prop. 5.15 they agree as sections of $\mathscr{F}$. Since the equality $\chi = \vartheta$ is none other than our formula (4.24), the proof of Prop. 4.4 is now complete.

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